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2006 J. Phys. A: Math. Gen. 39 4119

(http://iopscience.iop.org/0305-4470/39/16/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 4119–4145 doi:10.1088/0305-4470/39/16/002

Exact product form for the anisotropic simple cubic lattice Green function

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Received 13 November 2005
Published 31 March 2006
Online at stacks.iop.org/JPhysA/39/4119

Abstract
The analytical properties of the lattice Green function

$$G(2n,n,n;\alpha,w)=\frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\cos 2n\theta_1\cos n\theta_2\cos n\theta_3}{w-\alpha\cos\theta_1-\cos\theta_2-\cos\theta_3}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3$$

are investigated, where n is an integer, w is a complex variable and α is a real parameter in the interval $(0,\infty)$. In particular, it is shown that $G(2n,n,n;\alpha,w)$ is a solution of a fourth-order linear differential equation of the Fuchsian type. From this differential equation, it is proved that $G(2n,n,n;\alpha,w)$ can be expressed in the hypergeometric form

$$\begin{aligned}wG(2n,n,n;\alpha,w)&=\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}\left(\frac{w^2}{w^2+4-\alpha^2}\right)^{\frac{1}{2}}\\&\quad\times\left[\frac{w^2}{8\alpha}\left(\sqrt{1-\frac{(2-\alpha)^2}{w^2}}-\sqrt{1-\frac{(2+\alpha)^2}{w^2}}\right)^2\right]^{2n}\\&\quad\times{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\eta_+\right){}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\eta_-\right)\end{aligned}$$

where

$$\begin{aligned}\eta_\pm\equiv\eta_\pm(\alpha,w)&=\frac{1}{2}+\frac{w^2}{2\left(w^2+4-\alpha^2\right)^2}\left\{\pm16\sqrt{1-\frac{\alpha^2}{w^2}}\right.\\&\quad\left.-\left(w^2-4-\alpha^2\right)\sqrt{1-\frac{(2-\alpha)^2}{w^2}}\sqrt{1-\frac{(2+\alpha)^2}{w^2}}\right\}\end{aligned}$$

and $(\beta)_n$ denotes the Pochhammer symbol. This formula is valid for varying values of w in the neighbourhood of $w=\infty$, provided that the argument function $\eta_+(\alpha,w)$ does not take real values in the interval $(1,\infty)$. The ${}_2F_1$ product form is used to determine the asymptotic behaviour of $G(2n,n,n;\alpha,w)$ as $n\to\infty$. Finally, a five-term linear recurrence relation is given for $G(2n,n,n;\alpha,w)$.

PACS numbers: 02.30.Hq, 05.20.−y

1. Introduction

The lattice Green function

$$G(\ell, m, n; \alpha, w) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\cos \ell\theta_1 \cos m\theta_2 \cos n\theta_3}{w - \alpha \cos\theta_1 - \cos\theta_2 - \cos\theta_3} \, d\theta_1 \, d\theta_2 \, d\theta_3, \tag{1.1}$$

where $\{\ell, m, n\}$ is a set of integers, $w = w_1 + iw_2$ is a complex variable and α is a real nonzero parameter in the interval $(-\infty, \infty)$, plays an important role in many lattice statistical problems which involve the simple cubic lattice with partially anisotropic nearest-neighbour interactions (Berlin and Kac 1952, Duffin 1953, Maradudin *et al* 1960, Montroll and Weiss 1965, Joyce 1972a, Kobelev and Kolomeisky 2002). We shall assume, without loss of generality, that $\ell \geqslant 0$ and $m \geqslant n \geqslant 0$. It is readily seen from (1.1) that

$$G(\ell, m, n; -\alpha, w) = (-1)^\ell G(\ell, m, n; \alpha, w). \tag{1.2}$$

We shall, therefore, also restrict our attention to the case $\alpha \in (0, \infty)$.

The triple integral (1.1) defines a single-valued analytic function $G(\ell, m, n; \alpha, w)$ in the complex (w_1, w_2) plane provided that a cut is made along the real axis from $w = -2 - \alpha$ to $w = 2 + \alpha$. We shall denote the set of points (w_1, w_2) in this cut plane by $\mathcal{C}^-$. It is found from (1.1) that $G(\ell, m, n; \alpha, w)$ satisfies the symmetry relation

$$G(\ell, m, n; \alpha, -w) = (-1)^{\ell+m+n+1} G(\ell, m, n; \alpha, w). \tag{1.3}$$

We see, therefore, that it is only strictly necessary to analyse the properties of (1.1) for points $w \in \mathcal{C}^-$ which have $w_1 \geqslant 0$.

For many applications in solid-state physics (Koster and Slater 1954, Wolfram and Callaway 1963, Katsura *et al* 1971) one needs to know the limiting behaviour of $G(\ell, m, n; \alpha, w)$ as w approaches the upper and lower edges of the cut in the (w_1, w_2) plane. It is convenient, therefore, to introduce the definitions

$$\begin{aligned} G^\pm(\ell, m, n; \alpha, w_1) &\equiv \lim_{\epsilon \to 0+} G(\ell, m, n; \alpha, w_1 \pm i\epsilon) \\ &\equiv G_R(\ell, m, n; \alpha, w_1) \mp i G_I(\ell, m, n; \alpha, w_1) \end{aligned} \tag{1.4}$$

where $-2 - \alpha < w_1 < 2 + \alpha$. When $|w_1| \geqslant 2 + \alpha$, the imaginary part of $G^\pm(\ell, m, n; \alpha, w_1)$ is always equal to zero. Delves and Joyce (2001a) have proved that (1.4) can be written in the single integral form

$$G^\pm(\ell, m, n; \alpha, w_1) = (\mp i)^{\ell+m+n+1} \int_0^\infty \exp(\pm i w_1 t) J_\ell(\alpha t) J_m(t) J_n(t) \, dt \tag{1.5}$$

where $-2 - \alpha < w_1 < 2 + \alpha$ and $J_n(t)$ denotes a Bessel function of the first kind of order n. When $\ell + m + n$ is an even integer, it follows from (1.4) and (1.5) that

$$G_R(\ell, m, n; \alpha, w_1) = (-1)^{(\ell+m+n)/2} \int_0^\infty \sin(w_1 t) J_\ell(\alpha t) J_m(t) J_n(t) \, dt \tag{1.6}$$

$$G_I(\ell, m, n; \alpha, w_1) = (-1)^{(\ell+m+n)/2} \int_0^\infty \cos(w_1 t) J_\ell(\alpha t) J_m(t) J_n(t) \, dt. \tag{1.7}$$

Similar formulae can also be obtained when $\ell + m + n$ is an odd integer.

The first exact evaluation of the Green function (1.1) was carried out by Watson (1939) for the special case $\ell = m = n = 0$, $\alpha = 1$ and $w = 3$. In particular, he proved that

$$G(0, 0, 0; 1, 3) = \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right) \left\{ \frac{2}{\pi} K \left[\left(2 - \sqrt{3}\right) \left(\sqrt{3} - \sqrt{2}\right) \right] \right\}^2 \tag{1.8}$$

where $K(k)$ denotes the complete elliptic integral of the first kind with a modulus k. From the work of Horiguchi (1971), Joyce (1972b, 1973), Horiguchi and Morita (1975), Morita (1975), Glasser and Boersma (2000) and Joyce (2002), it is now known that the *isotropic* Green function $G(\ell, m, n; 1, w)$ can be expressed in terms of complete elliptic integrals of the first and second kinds at an *arbitrary* lattice point $\{\ell, m, n\}$ and at *any* point $w \in \mathcal{C}^-$. Recently, Joyce and Delves (2004a) have shown that the diagonal Green function $G(n, n, n; 1, w)$ can be written in terms of a simple product of two ${}_2F_1$ hypergeometric functions for all integer values of n, provided that w lies in a sufficiently small neighbourhood of $w = \infty$. A similar ${}_2F_1$ product form has also been derived by Joyce and Delves (2004b) for the Green function $G(2n, n, n; 1, w)$.

Montroll (1956) extended the method of Watson (1939) and obtained an exact formula for the *anisotropic* Green function $G(0, 0, 0; \alpha, 2+\alpha)$, where $\alpha \in (0, \infty)$. His final result can be written in the form

$$G(0,0,0;\alpha,2+\alpha) = \frac{\sqrt{2}}{\alpha}\left(\sqrt{2}\sqrt{1+\alpha} - \sqrt{2+\alpha}\right)\frac{4}{\pi^2}K(k_+)K(k_-) \tag{1.9}$$

where

$$k_\pm \equiv k_\pm(\alpha) = \frac{1}{\alpha}\left(\sqrt{2}\sqrt{1+\alpha} - \sqrt{2+\alpha}\right)\left(\sqrt{2+\alpha} \pm \sqrt{2}\right). \tag{1.10}$$

It has been shown by Delves and Joyce (2001a, 2001b) and Joyce *et al* (2003) that the Green function $G(0, 0, 0; \alpha, w)$ can also be evaluated in terms of a product of two complete elliptic integrals of the first kind for *all* $w \in \mathcal{C}^-$.

Our main aim in this paper is to generalize the work of Joyce and Delves (2004b) in order to determine the detailed analytic properties of the *anisotropic* lattice Green function $G(2n, n, n; \alpha, w)$, where $\alpha \in (0, \infty)$. In particular, it is proved in section 2 that $G(2n, n, n; \alpha, w)$ is a solution of a fourth-order differential equation of the Fuchsian type. In sections 3 and 4, this differential equation is used to show that $G(2n, n, n; \alpha, w)$ can be expressed in terms of a product of two ${}_2F_1$ hypergeometric functions. In section 5, it is demonstrated that the evaluation of $G(2n, n, n; \alpha, w)$ can be simplified when w takes certain special values. The asymptotic behaviour of $G(2n, n, n; \alpha, w)$ as $n \to \infty$ is established in section 6. Finally, in section 7 we give a five-term linear recurrence relation for $G(2n, n, n; \alpha, w)$.

2. Basic results for the Green function $G(2n, n, n; \alpha, w)$

In this section, we shall derive a series expansion for $G(2n, n, n; \alpha, w)$ about the point $w = \infty$. It will also be proved that $G(2n, n, n; \alpha, w)$ satisfies a fourth-order differential equation of the Fuchsian type.

2.1. Series expansion for $G(2n, n, n; \alpha, w)$ about $w = \infty$

We begin by applying the formula

$$\lambda^{-1} = \int_0^\infty \exp(-\lambda t)\,\mathrm{d}t, \tag{2.1}$$

where $\mathrm{Re}(\lambda) > 0$, to the integrand denominator in (1.1). The resulting multiple integral can then be simplified using the well-known result

$$\frac{1}{\pi}\int_0^\pi \cos(n\theta)\exp(t\cos\theta)\,\mathrm{d}\theta = I_n(t) \tag{2.2}$$

where $I_n(t)$ denotes a modified Bessel function of the first kind. In this manner, we find that

$$G(\ell, m, n; \alpha, w) = \int_0^\infty \exp(-wt) I_\ell(\alpha t) I_m(t) I_n(t)\, \mathrm{d}t \tag{2.3}$$

where $\mathrm{Re}(w) \geqslant 2+\alpha$.

We now consider the case $\ell = 2n$, $m = n$ and introduce the Taylor series expansion

$$I_{2n}(\alpha t)[I_n(t)]^2 = \frac{\alpha^{2n}}{(n!)^2(2n)!}\left(\frac{t}{2}\right)^{4n} \sum_{j=0}^{\infty} \beta_j(n,\alpha)\left(\frac{t}{2}\right)^{2j} \tag{2.4}$$

where $|t| < \infty$ and $\beta_0(n,\alpha) = 1$. A formula for the coefficient $\beta_j(n,\alpha)$ in (2.4) can be determined by considering the generating function identity

$$ {}_0F_1\big(-; 2n+1; \alpha^2 x\big)\big[{}_0F_1(-; n+1; x)\big]^2 \equiv \sum_{j=0}^{\infty} \beta_j(n,\alpha) x^j \tag{2.5}$$

where ${}_0F_1$ denotes a generalized hypergeometric series. If the standard relation (see Erdélyi *et al* (1953), p 185)

$$\big[{}_0F_1(-; n+1; x)\big]^2 = {}_1F_2\big(n+\tfrac{1}{2}; n+1, 2n+1; 4x\big) \tag{2.6}$$

is applied to (2.5), it is readily found that

$$\beta_j(n,\alpha) = \frac{\alpha^{2j}}{(2n+1)_j\, j!}\, {}_3F_2\left[\begin{matrix} -j, & -j-2n, & n+\frac{1}{2}; \\ n+1, & 2n+1; & \end{matrix}\; 4/\alpha^2\right] \tag{2.7}$$

where $(2n+1)_j$ is a Pochhammer symbol.

Finally, we substitute (2.4) into the integral representation (2.3), with $\ell = 2n$ and $m = n$. This procedure yields the required series expansion

$$wG(2n, n, n; \alpha, w) \equiv y_G(n; \alpha, z) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}(2\alpha z)^{2n} \sum_{j=0}^{\infty} \mu_j(n;\alpha) z^j \tag{2.8}$$

where $z = 1/w^2$, $|z| \leqslant 1/(2+\alpha)^2$ and

$$\mu_j(n;\alpha) = \big(2n+\tfrac{1}{2}\big)_j \frac{\alpha^{2j}}{j!}\, {}_3F_2\left[\begin{matrix} -j, -j-2n, & n+\frac{1}{2}; \\ n+1, & 2n+1; \end{matrix}\; 4/\alpha^2\right]. \tag{2.9}$$

When $n = 0$, formula (2.9) is in agreement with the earlier work of Delves and Joyce (2001a). From (2.9) one finds that

$$\mu_0(n;\alpha) = 1 \tag{2.10}$$

$$\mu_1(n;\alpha) = \frac{(4n+1)}{2(n+1)}\big[(2+\alpha^2) + (4+\alpha^2)n\big] \tag{2.11}$$

$$\mu_2(n;\alpha) = \frac{(4n+1)(4n+3)}{8(n+1)(n+2)}\big[2(6+8\alpha^2+\alpha^4) + (32+24\alpha^2+3\alpha^4)n + (4+\alpha^2)^2 n^2\big] \tag{2.12}$$

$$\begin{aligned}\mu_3(n;\alpha) = \frac{(4n+1)(4n+3)(4n+5)}{48(n+1)(n+2)(n+3)}\big[&6(20+54\alpha^2+18\alpha^4+\alpha^6) \\ &+ (368+540\alpha^2+162\alpha^4+11\alpha^6)n \\ &+ 6(4+\alpha^2)(12+9\alpha^2+\alpha^4)n^2 + (4+\alpha^2)^3 n^3\big].\end{aligned} \tag{2.13}$$

2.2. Fourth-order differential equation for $G(2n, n, n; \alpha, w)$

In order to derive a differential equation for $G(2n, n, n; \alpha, w)$, we generalize the method of Iwata (1979) and introduce the set of Green functions

$$\Big\{G \equiv G(2n, n, n; \alpha, w), G_1 \equiv G(2n+1, n, n; \alpha, w), G_2 \equiv G(2n, n+1, n; \alpha, w),$$
$$G_3 \equiv G(2n+1, n+1, n; \alpha, w), G_4 \equiv G(2n+1, n+1, n+1; \alpha, w),$$
$$G_5 \equiv G(2n, n+1, n+1; \alpha, w)\Big\}. \tag{2.14}$$

It is readily found by differentiating (2.3) with respect to w and then integrating the resulting expression by parts that

$$wG' = -(4n+1)G + \alpha G_1' + 2G_2' \tag{2.15}$$
$$wG_1' = \alpha G' + 2G_3' \tag{2.16}$$
$$wG_2' = -2nG_2 + G' + \alpha G_3' + G_5' \tag{2.17}$$
$$wG_3' = (2n+1)G_3 + G_1' + \alpha G_2' + G_4' \tag{2.18}$$
$$wG_4' = 2(2n+1)G_4 + 2G_3' + \alpha G_5' \tag{2.19}$$
$$wG_5' = G_5 + 2G_2' + \alpha G_4' \tag{2.20}$$

where the prime denotes the derivative with respect to w.

Careful systematic elimination of $\{G_j : j = 1, 2, \ldots, 5\}$ from equations (2.15)–(2.20) enables one to prove that $y_G(n; \alpha, z) \equiv wG(2n, n, n; \alpha, w)$ is a solution of the fourth-order linear differential equation

$$\mathbf{L}_4(y) \equiv \sum_{j=0}^{4} f_j(n; \alpha, z)\, \mathrm{D}^{4-j} y = 0 \tag{2.21}$$

where

$$\begin{aligned} f_0(n; \alpha, z) = {} & 8z^4(1-\alpha^2 z)\big[1-(2-\alpha)^2 z\big]\big[1-(2+\alpha)^2 z\big] \\ & \times\Big\{18n^2 - 3\big[(1-\alpha^2) + 4(4-\alpha^2)n^2\big]z \\ & - (4-\alpha^2)\big[5(1-\alpha^2) - 2(4-\alpha^2)n^2\big]z^2\Big\} \end{aligned} \tag{2.22}$$

$$\begin{aligned} f_1(n; \alpha, z) = 4z^3\Big\{ & 216n^2 - \big[30(1-\alpha^2) + 6(488 + 133\alpha^2)n^2\big]z \\ & + \big[25(1-\alpha^2)(8+7\alpha^2) + 8(1\,544 + 74\alpha^2 + 83\alpha^4)n^2\big]z^2 \\ & + \big[5(1-\alpha^2)(224 + 52\alpha^2 - 75\alpha^4) \\ & - 4(4-\alpha^2)(1\,168 + 512\alpha^2 + 39\alpha^4)n^2\big]z^3 \\ & - (4-\alpha^2)\big[15(1-\alpha^2)(96 - 20\alpha^2 + 23\alpha^4) \\ & - 48(4-\alpha^2)(12 + 25\alpha^2 - 4\alpha^4)n^2\big]z^4 \\ & + 23\alpha^2(4-\alpha^2)^3\big[5(1-\alpha^2) - 2(4-\alpha^2)n^2\big]z^5\Big\} \end{aligned} \tag{2.23}$$

$$\begin{aligned} f_2(n; \alpha, z) = 2z^2\Big\{ & 72n^2(7-5n^2) - 12\big[4(1-\alpha^2) \\ & + (847 + 296\alpha^2)n^2 - 16(8+\alpha^2)n^4\big]z \end{aligned}$$

$$
\begin{aligned}
&+\left[2\left(1-\alpha^2\right)\left(544+233\alpha^2\right)+2\left(26\,392+778\alpha^2+2\,611\alpha^4\right)n^2\right.\\
&-16\left(4-\alpha^2\right)\left(52+11\alpha^2\right)n^4\big]z^2+\left[\left(1-\alpha^2\right)\left(448+608\alpha^2-1\,365\alpha^4\right)\right.\\
&-2\left(4-\alpha^2\right)\left(10\,072+7\,142\alpha^2-237\alpha^4\right)n^2+256\left(4-\alpha^2\right)^2n^4\big]z^3\\
&-2\left[\left(1-\alpha^2\right)\left(4-\alpha^2\right)\left(3\,040-974\alpha^2+811\alpha^4\right)\right.\\
&-\left(4-\alpha^2\right)^2\left(1\,256+3\,758\alpha^2-715\alpha^4\right)n^2+4\left(4-\alpha^2\right)^4n^4\big]z^4\\
&+135\alpha^2\left(4-\alpha^2\right)^3\left[5\left(1-\alpha^2\right)-2\left(4-\alpha^2\right)n^2\right]z^5\Big\}
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
f_3(n;\alpha,z)=3z\Big\{&48n^2\left(1-5n^2\right)-16n^2\left[\left(149+64\alpha^2\right)-40\left(2+\alpha^2\right)n^2\right]z\\
&+8\left[3\left(1-\alpha^2\right)\left(11+3\alpha^2\right)+\left(2\,044+65\alpha^2+363\alpha^4\right)n^2\right.\\
&-8\left(4-\alpha^2\right)\left(20+3\alpha^2\right)n^4\big]z^2-2\left[\left(1-\alpha^2\right)\left(736+8\alpha^2+177\alpha^4\right)\right.\\
&+\left(4-\alpha^2\right)(2\,880+4\,444\alpha^2-649\alpha^4)n^2-32\left(4-\alpha^2\right)^2\left(8-\alpha^2\right)n^4\big]z^3\\
&-\left(4-\alpha^2\right)\Big[\left(1-\alpha^2\right)\left(1\,760-1\,076\alpha^2+589\alpha^4\right)\\
&-4\left(4-\alpha^2\right)\left(216+938\alpha^2-205\alpha^4\right)n^2+16\left(4-\alpha^2\right)^3n^4\Big]z^4\\
&+65\alpha^2\left(4-\alpha^2\right)^3\left[5\left(1-\alpha^2\right)-2\left(4-\alpha^2\right)n^2\right]z^5\Big\}
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
f_4(n;\alpha,z)=576n^6&-24n^2\left[3\left(2+\alpha^2\right)+\left(8-35\alpha^2\right)n^2+16\left(4-\alpha^2\right)n^4\right]z\\
&+4n^2\left[\left(532-38\alpha^2+181\alpha^4\right)-\left(4-\alpha^2\right)\left(404-215\alpha^2\right)n^2\right.\\
&+16\left(4-\alpha^2\right)^2n^4\big]z^2-6\left[6\left(1-\alpha^2\right)\left(10+\alpha^2+\alpha^4\right)\right.\\
&+\left(4-\alpha^2\right)\left(4+511\alpha^2-116\alpha^4\right)n^2-16\left(4-\alpha^2\right)^3n^4\big]z^3\\
&-3\left(4-\alpha^2\right)\left[\left(1-\alpha^2\right)\left(40-104\alpha^2+31\alpha^4\right)\right.\\
&-2\left(4-\alpha^2\right)\left(28+139\alpha^2-35\alpha^4\right)n^2+4\left(4-\alpha^2\right)^3n^4\big]z^4\\
&+15\alpha^2\left(4-\alpha^2\right)^3\left[5\left(1-\alpha^2\right)-2\left(4-\alpha^2\right)n^2\right]z^5
\end{aligned}
\tag{2.26}
$$

where $D\equiv d/dz$ and $z=1/w^2$. For the special case $n=0$, the differential equation $\mathbf{L}_4(y)=0$ is consistent with the work of Delves and Joyce (2001a, equations (2.27)–(2.32)).

2.3. *Singularity structure of the differential equation (2.21)*

The basic differential equation (2.21) is of the Fuchsian type and has, in general, seven regular singular points at $z=0$,

$$z_1=1/(2+\alpha)^2 \tag{2.27}$$

$$z_2=1/\alpha^2 \tag{2.28}$$

$$z_3=1/(2-\alpha)^2 \tag{2.29}$$

$$z_4^{\pm}=\frac{3\left\{\left(\alpha^2-1\right)-4\left(4-\alpha^2\right)n^2\pm\sqrt{\left(1-\alpha^2\right)\left[\left(1-\alpha^2\right)+48\left(4-\alpha^2\right)n^2\right]}\right\}}{2\left(4-\alpha^2\right)\left[5\left(1-\alpha^2\right)-2\left(4-\alpha^2\right)n^2\right]} \tag{2.30}$$

and $z = \infty$. The Riemann P-symbol (see Ince (1927), p 370) associated with equation (2.21) is given by

$$P\begin{bmatrix} 0 & z_1 & z_2 & z_3 & z_4^+ & z_4^- & \infty & \\ 2n & 0 & 0 & 0 & 0 & 0 & 1 & \\ -2n & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & z \\ n & 2 & 2 & 2 & 2 & 2 & \frac{3}{2} & \\ -n & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 4 & 4 & \frac{5}{2} & \end{bmatrix}. \tag{2.31}$$

In this scheme, the singular points are placed on the first row with the roots of the corresponding indicial equations beneath them. For an arbitrary Nth-order Fuchsian equation with ν regular singular points in the finite z plane and a regular singular point at $z = \infty$, it can be shown (Ince 1927, p 371) that the sum of all the exponents in the Riemannian scheme is an *invariant* equal to $\frac{1}{2}N(N-1)(\nu-1)$. We see directly from (2.31) that the differential equation (2.21) has the correct Fuchsian invariant of 30 . It is also clear from the P-symbol that the expansion (2.8) will give a series solution of $\mathbf{L}_4(y) = 0$ which is associated with the exponent $2n$ at $z = 0$.

The singular points $z_4^{\pm}$ are of interest because the *general* solution of $\mathbf{L}_4(y) = 0$ is *analytic* at these points. This unusual type of singularity is known as an *apparent* (or *accidental*) singularity (Ince 1927, p 406). It is interesting to note that Zenine *et al* (2004) have shown that Fuchsian differential equations with apparent singularities also play an important role in the analysis of the n-particle contribution to the zero-field susceptibility of the square lattice Ising model.

Finally, it should be pointed out that for special values of (α, n) at least two of the singular points $z = 0, z_1, z_2, z_3, z_4^{\pm}$ and ∞ are coincident. Under these circumstances, it is necessary to modify the structure of the general P-symbol (2.31).

3. Analysis of the differential equation $\mathbf{L}_4(y) = 0$

Our main aim in this section is to prove that any solution of $\mathbf{L}_4(y) = 0$ can be expressed in terms of a product of solutions of two second-order differential equations in normal form. We shall also show that these differential equations can be transformed into a standard hypergeometric form.

3.1. Product solutions for the differential equation (2.21)

It can be established by following a method recently described by Delves and Joyce (2001a, pp 81–4) that any solution of the differential equation (2.21) can be expressed in the product form

$$y(n;\alpha,z) = M(\alpha,z)H_+(n;\alpha,z)H_-(n;\alpha,z) \tag{3.1}$$

where

$$M(\alpha,z) = \frac{1}{z}\left[\prod_{i=1}^{3}\left(1-\frac{z}{z_i}\right)^{-1/2}\right]\left(1-\frac{z}{z_5}\right)^{1/2} \tag{3.2}$$

$\{z_i : i = 1, 2, 3\}$ are the regular singular points defined in (2.27)–(2.29),

$$z_5 = 3/\left(4-\alpha^2\right) \tag{3.3}$$

and $H_+(n;\alpha,z)$, $H_-(n;\alpha,z)$ are appropriate solutions of the second-order differential equations

$$\left[\mathrm{D}^2 + U_+(n;\alpha,z)\right]H = 0 \tag{3.4}$$

$$\left[\mathrm{D}^2 + U_-(n;\alpha,z)\right]H = 0, \tag{3.5}$$

respectively, with $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. We note that the introduced singularity z_5 is the same as that used by Delves and Joyce (2001a) in the analysis of $G(0,0,0;\alpha,w)$. The coefficients $U_\pm(n;\alpha,z)$ in (3.4) and (3.5) can be expressed in the form

$$U_\pm(n;\alpha,z) = U_\pm^{(0)}(\alpha,z) - U_\pm^{(2)}(\alpha,z)n^2 \tag{3.6}$$

where

$$\begin{aligned}U_\pm^{(0)}(\alpha,z) = {} & \frac{7(2+\alpha^2)}{12z} + \frac{1}{4z^2} + \frac{\alpha^4(16-19\alpha^2)}{64(1-\alpha^2)(1-\alpha^2 z)} + \frac{3\alpha^4}{16(1-\alpha^2 z)^2} \\ & + \frac{(2-\alpha)^4(8+32\alpha-23\alpha^2-5\alpha^3)}{128\alpha(1-\alpha)\left[1-(2-\alpha)^2 z\right]} + \frac{3(2-\alpha)^4}{16\left[1-(2-\alpha)^2 z\right]^2} \\ & - \frac{(2+\alpha)^4(8-32\alpha-23\alpha^2+5\alpha^3)}{128\alpha(1+\alpha)\left[1-(2+\alpha)^2 z\right]} + \frac{3(2+\alpha)^4}{16\left[1-(2+\alpha)^2 z\right]^2} \\ & - \frac{(4-\alpha^2)^2(40-31\alpha^2)}{192(1-\alpha^2)\left[3-(4-\alpha^2)z\right]} - \frac{3(4-\alpha^2)^2}{8\left[3-(4-\alpha^2)z\right]^2} \\ & \pm \frac{(\alpha^2-1)\left[3+5(4-\alpha^2)z\right]}{2z\left[3-(4-\alpha^2)z\right]^2\sqrt{1-(2+\alpha)^2 z}\,\sqrt{1-\alpha^2 z}\,\sqrt{1-(2-\alpha)^2 z}}\end{aligned} \tag{3.7}$$

and

$$U_\pm^{(2)}(\alpha,z) = \left[\frac{1}{z\sqrt{1-(2+\alpha)^2 z}\,\sqrt{1-(2-\alpha)^2 z}} \mp \frac{1}{2z\sqrt{1-\alpha^2 z}}\right]^2. \tag{3.8}$$

3.2. Transformation of (3.4) and (3.5) to hypergeometric form

We begin the analysis by considering the hypergeometric equation (Erdélyi *et al* 1953, p 56)

$$s(1-s)\frac{\mathrm{d}^2 Y}{\mathrm{d}s^2} + [c-(a+b+1)s]\frac{\mathrm{d}Y}{\mathrm{d}s} - abY = 0. \tag{3.9}$$

Next, we apply the transformation

$$Y = s^{-c/2}(1-s)^{-\frac{1}{2}(1+a+b-c)}Y_N \tag{3.10}$$

to (3.9). This procedure leads to the normal form

$$\frac{\mathrm{d}^2 Y_N}{\mathrm{d}s^2} + \Omega(s)Y_N = 0 \tag{3.11}$$

where

$$\Omega(s) = \frac{c(2-c) - 2\left[2ab + c(1-a-b)\right]s + \left[1-(a-b)^2\right]s^2}{4s^2(1-s)^2}. \tag{3.12}$$

A general transformation $s = s(z)$ is now applied to the independent variable in equation (3.11). Hence, we find that

$$\frac{\mathrm{d}^2 Y_N}{\mathrm{d}z^2} - \frac{s''}{s'}\frac{\mathrm{d}Y_N}{\mathrm{d}z} + (s')^2\Omega(s)Y_N = 0 \tag{3.13}$$

where s' denotes the derivative of $s(z)$ with respect to z. Finally, the application of the transformation

$$Y_N = (s')^{1/2} H \tag{3.14}$$

brings (3.13) back to the further normal form

$$\frac{d^2 H}{dz^2} + \left[\frac{1}{2}\{s, z\} + (s')^2 \Omega(s)\right] H = 0 \tag{3.15}$$

where

$$\{s, z\} \equiv \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 \tag{3.16}$$

is the Schwarzian derivative of $s(z)$ with respect to z.

From the work of Delves and Joyce (2001b) we are motivated to take $a = (4n+1)/8$, $b = (4n+3)/8$, $c = n+1$, and consider two particular transformations $s = s_\pm(z)$ for which equations (3.15) are identical to the differential equations (3.4) and (3.5), respectively. Hence, it is seen that $s_\pm(z)$ must satisfy the nonlinear equation

$$\frac{1}{2}\{s_\pm, z\} + \left(s'_\pm\right)^2 \Omega(s_\pm) = U_\pm^{(0)}(\alpha, z) - U_\pm^{(2)}(\alpha, z) n^2 \tag{3.17}$$

where $U_\pm^{(0)}(\alpha, z)$, $U_\pm^{(2)}(\alpha, z)$ are defined in (3.7) and (3.8), respectively, and

$$\Omega(s) = \frac{\left[\left(16 - 19s + 15s^2\right) - 16(1-s)n^2\right]}{64s^2(1-s)^2}. \tag{3.18}$$

We shall now consider the possibility of α-dependent transformations $s_\pm(z) \equiv s_\pm(\alpha, z)$ which are *independent* of the variable n. From (3.17), it is evident that transformations of this type must satisfy simultaneously the *two* nonlinear equations

$$\frac{1}{2}\{s_\pm, z\} + \left(s'_\pm\right)^2 \frac{\left(16 - 19s_\pm + 15s_\pm^2\right)}{64s_\pm^2(1-s_\pm)^2} = U_\pm^{(0)}(\alpha, z) \tag{3.19}$$

and

$$\frac{\left(s'_\pm\right)^2}{4s_\pm^2(1-s_\pm)} = U_\pm^{(2)}(\alpha, z). \tag{3.20}$$

It is found that the *general* solution of the simpler equation (3.20) can be written in the form

$$s_\pm(\alpha, z; C_\pm) = 4C_\pm X_\pm / (1 + C_\pm X_\pm)^2 \tag{3.21}$$

where

$$X_\pm \equiv X_\pm(\alpha, z) = \frac{64z\left[1 \pm \sqrt{1-\alpha^2 z}\right]^2}{\left[\sqrt{1-(2-\alpha)^2 z} + \sqrt{1-(2+\alpha)^2 z}\right]^4} \tag{3.22}$$

and $C_\pm$ are arbitrary constants. We find that the algebraic solution (3.21) satisfies a quartic equation of the type

$$\sum_{j=0}^{4} p_j(\alpha, z; C_\pm)\left[s_\pm(\alpha, z; C_\pm)\right]^{4-j} = 0 \tag{3.23}$$

where $\{p_j(\alpha, z; C_\pm) : j = 0, 1, 2, 3, 4\}$ are polynomials in $\{\alpha, z, C_\pm\}$. We have verified by direct substitution that (3.21) is *also* a solution of the Schwarzian equation (3.19), *provided* *that* $C_\pm \equiv 1$. We see, therefore, that the required transformation functions $s_\pm(\alpha, z)$ are given by $s_\pm(\alpha, z; 1)$, respectively. When $C_\pm = 1$, the left-hand side of (3.23) becomes the *square* of a quadratic polynomial and as a result $s_\pm(\alpha, z; 1)$ can be expressed in the simplified form

$$s_\pm(\alpha, z) \equiv s_\pm(\alpha, z; 1) = \frac{16z}{\left[1 + \left(4 - \alpha^2\right)z\right]^4}\Big[1 - \left(4 + \alpha^2\right)z \pm \sqrt{1 - (2 + \alpha)^2 z}\,\sqrt{1 - \alpha^2 z}\,\sqrt{1 - (2 - \alpha)^2 z}\Big]^2. \tag{3.24}$$

Finally, we note that for the special case $n = 0$ the simple differential equation (3.20) is no longer available and one has to deal directly with the more difficult Schwarzian equation (3.19) (see Delves and Joyce (2001a, 2001b)).

3.3. *Transformation formulae for* $H_\pm(n; \alpha, z)$

It follows from (3.10), (3.14), with $\{a = (4n + 1)/8, b = (4n + 3)/8, c = n + 1\}$, and the Schwarzian transformation theory developed above that the solutions $H_+(n; \alpha, z)$ and $H_-(n; \alpha, z)$ of the second-order differential equations (3.4) and (3.5), respectively, can be written in the form

$$H_+(n; \alpha, z) = (s_+)^{(n+1)/2}(1 - s_+)^{1/4}\left(s'_+\right)^{-1/2} Y_1(s_+) \tag{3.25}$$

$$H_-(n; \alpha, z) = (s_-)^{(n+1)/2}(1 - s_-)^{1/4}\left(s'_-\right)^{-1/2} Y_2(s_-). \tag{3.26}$$

In these equations, $\{Y_j(s) : j = 1, 2\}$ are two (not necessarily independent) solutions of the hypergeometric equation (3.9), with $\{a = (4n + 1)/8, b = (4n + 3)/8, c = n + 1\}$, and the transformation functions $s_\pm = s_\pm(\alpha, z)$ are defined in equation (3.24). In the next section, we shall use the results (3.25) and (3.26) to evaluate the Green function $G(2n, n, n; \alpha, w)$ in terms of ${}_2F_1$ hypergeometric functions.

4. Product formulae for the Green function $G(2n, n, n; \alpha, w)$

The main purpose in this section is to express the Green function $G(2n, n, n; \alpha, w)$ in terms of products of two ${}_2F_1$ hypergeometric functions.

4.1. *Derivation of the basic* ${}_2F_1$ *product formula for* $G(2n, n, n; \alpha, w)$

We begin by applying the transformation formulae (3.25) and (3.26) to equation (3.1). This procedure yields

$$y_G(n; \alpha, z) \equiv wG(2n, n, n; \alpha, w) = M(\alpha, z)(s_+s_-)^{(n+1)/2}[(1 - s_+)(1 - s_-)]^{1/4} \times \left(s'_+s'_-\right)^{-1/2} Y_1(s_+)Y_2(s_-) \tag{4.1}$$

where the multiplier $M(\alpha, z)$ is defined in (3.2) and $\{Y_j(s) : j = 1, 2\}$ are appropriate solutions of the hypergeometric equation (3.9), with $a = (4n + 1)/8$, $b = (4n + 3)/8$ and $c = n + 1$. If we substitute (3.24) into (4.1) it is found that

$$y_G(n; \alpha, z) = \frac{1}{\sqrt{3}}(4\alpha z)^{2n}\left[1 + \left(4 - \alpha^2\right)z\right]^{-2n-\frac{1}{2}} Y_1(s_+)Y_2(s_-). \tag{4.2}$$

In the neighbourhood of $z = 0$, formula (4.2) must be consistent with the expansion (2.8). This requirement can be satisfied by taking the solutions of the hypergeometric equation (3.9) to be

$$Y_1(s) = Y_2(s) = 3^{1/4}\sqrt{\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2^n n!}}\; {}_2F_1\left(\frac{n}{2} + \frac{1}{8}, \frac{n}{2} + \frac{3}{8}; n + 1; s\right). \tag{4.3}$$

From (4.2) and (4.3), we obtain the fundamental product formula

$$wG(2n, n, n; \alpha, w) = \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^2} (2\alpha z)^{2n} \left[1 + \left(4 - \alpha^2\right) z\right]^{-2n-\frac{1}{2}} \times {}_2F_1\left(\frac{n}{2} + \frac{1}{8}, \frac{n}{2} + \frac{3}{8}; n+1; s_+\right) {}_2F_1\left(\frac{n}{2} + \frac{1}{8}, \frac{n}{2} + \frac{3}{8}; n+1; s_-\right) \tag{4.4}$$

where the transformation functions $s_\pm = s_\pm(\alpha, z)$ are defined in equation (3.24) and $z = 1/w^2$. We have used (4.4) to generate the coefficients $\{\mu_j(n; \alpha) : 0 \leqslant j \leqslant 10\}$ in the expansion (2.8) and agreement was found with (2.9)–(2.13). It has also been verified by direct substitution that the product form (4.4) satisfies the Fuchsian differential equation (2.21). When $n = 0$, formula (4.4) reduces to a known result for $wG(0, 0, 0; \alpha, w)$ (see Delves and Joyce (2001a), equation (5.22)).

The region of validity in the z plane for formula (4.4) can be determined by first constructing a complex curve $\mathcal{C}(\alpha)$ which consists of the set of points

$$\{z : s_+(\alpha, z) \in [1, \infty)\}. \tag{4.5}$$

When $0 < \alpha < 2$, the curve $\mathcal{C}(\alpha)$ is closed with a self-intersection point at $z_7 = 1/(\alpha^2 - 4)$ and it divides the z plane into three regions $\mathcal{R}(\alpha)$, $\mathcal{R}_1(\alpha)$ and $\mathcal{R}_2(\alpha)$, where $\mathcal{R}(\alpha)$ contains the point $z = 0$ and $\mathcal{R}_1(\alpha)$ has a common boundary with $\mathcal{R}(\alpha)$. For the case $\alpha \geqslant 2$, the curve $\mathcal{C}(\alpha)$ is a simple curve which divides the z plane into two regions $\mathcal{R}(\alpha)$ and $\mathcal{R}_1(\alpha)$. In both cases, it is found that (4.4) is valid for all points $z \in \mathcal{R}(\alpha)$. When z is real and positive, formula (4.4) is valid provided that $z \leqslant z_8$, where

$$z_8 = \frac{\left(\sqrt{8 + \alpha^2} - 2\right)^2}{\left(4 + \alpha^2\right)^2}. \tag{4.6}$$

Because $z_8 < z_1$ for all $\alpha \in (0, \infty)$, it is evident that (4.4) has a very *limited* range of validity for real positive values of z.

If $z \in \mathcal{R}_1(\alpha)$ and is sufficiently close to the boundary between $\mathcal{R}(\alpha)$ and $\mathcal{R}_1(\alpha)$ then we can modify (4.4) by replacing the hypergeometric function ${}_2F_1(s_+)$ with a standard analytic continuation formula (Erdélyi *et al* 1953, p 108, equation (1)). In this manner, we obtain

$$wG(2n, n, n; \alpha, w) = \frac{1}{(2\pi)^{3/2}} \frac{(4\alpha z)^{2n}}{n!} \left[1 + \left(4 - \alpha^2\right) z\right]^{-2n-\frac{1}{2}} {}_2F_1\left(\frac{n}{2} + \frac{1}{8}, \frac{n}{2} + \frac{3}{8}; n+1; s_-\right) \times \left[\Gamma\left(\frac{n}{2} + \frac{1}{8}\right) \Gamma\left(\frac{n}{2} + \frac{3}{8}\right) {}_2F_1\left(\frac{n}{2} + \frac{1}{8}, \frac{n}{2} + \frac{3}{8}; \frac{1}{2}; 1 - s_+\right) + 2\Gamma\left(\frac{n}{2} + \frac{5}{8}\right) \Gamma\left(\frac{n}{2} + \frac{7}{8}\right) (1 - s_+)^{\frac{1}{2}} {}_2F_1\left(\frac{n}{2} + \frac{5}{8}, \frac{n}{2} + \frac{7}{8}; \frac{3}{2}; 1 - s_+\right)\right] \tag{4.7}$$

where $s_\pm = s_\pm(\alpha, z)$ are defined in equation (3.24). We find that (4.7) may be used to determine the value of $G(2n, n, n; \alpha, w)$ in the neighbourhood of the singular point $w = 2 + \alpha$. For example, when $n = 1000$, $\alpha = 3$ and $w = 5$, we find that

$$G(2000, 1000, 1000; 3, 5) = 0.000\,050\,329\,210\,405\,875\,431\,142\,404\,751\,449\,075\,836\,899\,570\,102\,268\,653\,400\,487\,847\,085\,989\,549\,801\,745\,308\,106\,454\,326\,426\,732\,270\,274\,107\,178\,728\,995\ldots. \tag{4.8}$$

Further analytic continuations of (4.4) can be constructed by generalizing the methods used by Delves and Joyce (2001a) for the case $n = 0$.

4.2. Alternative ${}_2F_1$ product formulae for $G(2n, n, n; \alpha, w)$

Next, we apply the Goursat quadratic transformation (Erdélyi *et al* 1953, p 112, equation (22))

$$ {}_2F_1\left[\frac{n}{2}+\frac{1}{8}, \frac{n}{2}+\frac{3}{8}; n+1; s\right] = (1-\eta)^{-n}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta\right), \tag{4.9} $$

where

$$ \eta = \frac{1}{2} - \frac{1}{2}\sqrt{1-s}, \tag{4.10} $$

to both ${}_2F_1$ functions in (4.4). After some algebraic simplifications, we obtain the alternative product form

$$ \begin{aligned} wG(2n, n, n; \alpha, w) = {}& \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^2}\left[1+\left(4-\alpha^2\right)z\right]^{-\frac{1}{2}} \\ & \times \left[\frac{1}{8\alpha z}\left(\sqrt{1-(2-\alpha)^2 z} - \sqrt{1-(2+\alpha)^2 z}\right)^2\right]^{2n} \\ & \times {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta_+\right) {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta_-\right) \end{aligned} \tag{4.11} $$

where

$$ \begin{aligned} \eta_\pm \equiv \eta_\pm(\alpha, w) = {}& \frac{1}{2} + \frac{1}{2\left[1+\left(4-\alpha^2\right)z\right]^2}\left\{\pm 16z\sqrt{1-\alpha^2 z}\right. \\ & \left. - \left[1-\left(4+\alpha^2\right)z\right]\sqrt{1-(2-\alpha)^2 z}\,\sqrt{1-(2+\alpha)^2 z}\right\} \end{aligned} \tag{4.12} $$

and $z = 1/w^2$. Formula (4.11) is valid for varying values of w in the neighbourhood of $w = \infty$, provided that the function $\eta_+(\alpha, w)$ does not take real values in the interval $(1, \infty)$. For the special case $\alpha = 1$, the product form (4.11) reduces to a known result for $wG(2n, n, n; 1, w)$ (see Joyce and Delves (2004b), equation (6.8)).

In order to establish the precise region of validity for (4.11), we first determine the set of points $\mathcal{S}$ in the w plane which give real values of $\eta_+(\alpha, w) \in \left(\frac{1}{2} + \frac{1}{4}\sqrt{4+\alpha^2}, +\infty\right)$. When $0 < \alpha < 2$, it is found that the set $\mathcal{S}$ forms a closed path which divides the w plane into two regions $\mathcal{D}(\alpha)$ and $\mathcal{D}_1(\alpha)$, as shown in figures 1(*a*)–(*c*). For the case $2 \leqslant \alpha < \infty$, the set $\mathcal{S}$ divides the w plane into three separate regions $\mathcal{D}(\alpha)$, $\mathcal{D}_1^\pm(\alpha)$, as shown in figures 1(*d*)–(*f*). The region $\mathcal{D}_1(\alpha)$ includes points on the real axis which are in the interval $\left(-\sqrt{4+\alpha^2}, \sqrt{4+\alpha^2}\right)$, while the regions $\mathcal{D}_1^+(\alpha)$ and $\mathcal{D}_1^-(\alpha)$ include the real intervals $\left(\sqrt{\alpha^2-4}, \sqrt{4+\alpha^2}\right)$ and $\left(-\sqrt{4+\alpha^2}, -\sqrt{\alpha^2-4}\right)$, respectively. From these results, it follows that (4.11) is valid for all $\alpha \in (0, \infty)$ provided that w lies in the *outer* region $\mathcal{D}(\alpha)$ of the cut w plane.

When $\alpha \in [2, \infty)$, and $w \in \mathcal{C}^-$ lies in one of the *inner* regions $\mathcal{D}_1^\pm(\alpha)$, it is necessary to modify the product form (4.11) by replacing the first ${}_2F_1$ function with a suitable analytic continuation formula. This procedure yields the alternative representation

$$ \begin{aligned} wG(2n, n, n; \alpha, w) = {}& \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^2}\left[1+\left(4-\alpha^2\right)z\right]^{-\frac{1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta_-\right) \\ & \times \left\{\left[\frac{1}{8\alpha z}\left(\sqrt{1-(2-\alpha)^2 z} - \sqrt{1-(2+\alpha)^2 z}\right)^2\right]^{2n} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta_+\right)\right. \\ & \left. \pm \mathrm{i}\sqrt{2}\left[-\frac{1}{\alpha^2 z}\left(1-\sqrt{1-\alpha^2 z}\right)^2\right]^n {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; 1-\eta_+\right)\right\} \end{aligned} \tag{4.13} $$

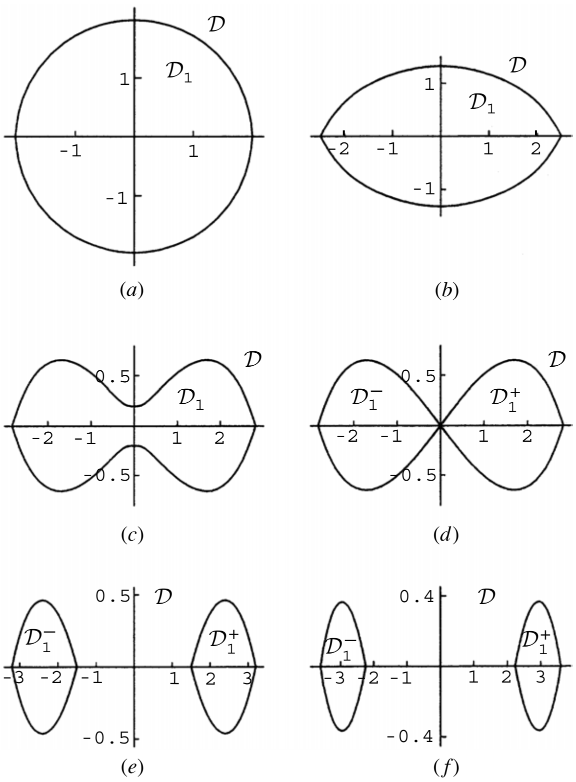

Figure 1. Regions $\mathcal{D} \equiv \mathcal{D}(\alpha)$, $\mathcal{D}_1 \equiv \mathcal{D}_1(\alpha)$, $\mathcal{D}_1^{\pm} \equiv \mathcal{D}_1^{\pm}(\alpha)$ in the w plane for various values of $\alpha \in (0, \infty)$. (*a*) $\alpha = 1/4$, (*b*) $\alpha = 3/2$, (*c*) $\alpha = 199/100$, (*d*) $\alpha = 2$, (*e*) $\alpha = 5/2$ and (*f*) $\alpha = 3$.

where $z = 1/w^2$ and $\eta_{\pm} \equiv \eta_{\pm}(\alpha, w)$ are given by (4.12). The upper positive sign in (4.13) is valid when $\{\mathrm{Re}(w) > 0, \mathrm{Im}(w) < 0\}$ and $\{\mathrm{Re}(w) < 0, \mathrm{Im}(w) > 0\}$, while the lower negative sign is valid when $\{\mathrm{Re}(w) > 0, \mathrm{Im}(w) > 0\}$ and $\{\mathrm{Re}(w) < 0, \mathrm{Im}(w) < 0\}$. Formula (4.13) is also valid when $\alpha \in (0, 2)$ and $w \in \mathcal{C}^-$ lie in the region $\mathcal{D}_1(\alpha)$, provided that $\mathrm{Re}(w) \neq 0$.

Difficulties arise when $\mathrm{Re}(w) = 0$ because the argument $1 - \eta_+$ of the third ${}_2F_1$ function in (4.13) takes a real value in the interval $(1, \infty)$, and as a result formula (4.13) gives an ambiguous value. In order to deal with this special case, we first make the substitution $w = \mathrm{i}w_2 + \delta$ into (4.13) and then take the limit $\delta \to 0+$. Next, we apply the identity

$$\left[\frac{w_2^2}{8\alpha}\left(\sqrt{\frac{(2-\alpha)^2}{w_2^2}+1} - \sqrt{\frac{(2+\alpha)^2}{w_2^2}+1}\right)^2\right]^{2n} {}_2F_1\left[\frac{1}{4}, \frac{3}{4}; n+1; \eta_+(\alpha, \mathrm{i}w_2)\right]$$
$$= \pm\sqrt{2}\left[\frac{w_2^2}{\alpha^2}\left(1 - \sqrt{\frac{\alpha^2}{w_2^2}+1}\right)^2\right]^n$$
$$\times \mathrm{Im}\left\{\lim_{\delta\to 0+} {}_2F_1\left[\frac{1}{4}, \frac{3}{4}; n+1; 1 - \eta_+(\alpha, \mathrm{i}w_2 + \delta)\right]\right\} \qquad (4.14)$$

where $\alpha \in (0, 2)$. The upper positive sign in (4.14) is valid for $-\sqrt{4-\alpha^2} < w_2 < 0$, while the lower negative sign is valid for $0 < w_2 < \sqrt{4-\alpha^2}$. Hence, we obtain the simplified product form

$$G(2n,n,n;\alpha,\mathrm{i}w_2) = \pm\mathrm{i}\sqrt{2}\,\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}\left(\frac{1}{4-\alpha^2-w_2^2}\right)^{1/2}\left[\frac{w_2^2}{\alpha^2}\left(1-\sqrt{1+\frac{\alpha^2}{w_2^2}}\right)^2\right]^n$$
$$\times\,{}_2F_1\left[\frac{1}{4},\frac{3}{4};n+1;\eta_-(\alpha,\mathrm{i}w_2)\right]\mathrm{Re}\left\{{}_2F_1\left[\frac{1}{4},\frac{3}{4};n+1;1-\eta_+(\alpha,\mathrm{i}w_2)\right]\right\} \tag{4.15}$$

where $\alpha\in(0,2)$. The role of the $\pm$ signs in (4.15) is the same as in equation (4.14).

We now apply the further quadratic transformation (Erdélyi *et al* 1953, p 112, equation (16))

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\eta\right) = (1-\eta)^{-1/4}\,{}_2F_1\left(\frac{1}{2},2n+\frac{1}{2};n+1;k^2\right), \tag{4.16}$$

where

$$k^2 = \frac{1}{2}-\frac{1}{2}(1-\eta)^{-1/2}, \tag{4.17}$$

to both ${}_2F_1$ functions in (4.11). After algebraic manipulation, we eventually obtain

$$wG(2n,n,n;\alpha,w) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}(2\alpha z)^{2n}$$
$$\times\left(\frac{1}{2}\sqrt{1-(2-\alpha)^2 z}+\frac{1}{2}\sqrt{1-(2+\alpha)^2 z}\right)^{-4n-1}$$
$$\times\,{}_2F_1\left(\frac{1}{2},2n+\frac{1}{2};n+1;k_+^2\right){}_2F_1\left(\frac{1}{2},2n+\frac{1}{2};n+1;k_-^2\right) \tag{4.18}$$

where

$$k_\pm^2 \equiv k_\pm^2(\alpha,z) = \frac{1}{2}-\frac{1}{2}\left[\sqrt{1-(2-\alpha)^2 z}+\sqrt{1-(2+\alpha)^2 z}\right]^{-3}$$
$$\times\left[\sqrt{1+(2-\alpha)\sqrt{z}}\,\sqrt{1-(2+\alpha)\sqrt{z}}+\sqrt{1-(2-\alpha)\sqrt{z}}\,\sqrt{1+(2+\alpha)\sqrt{z}}\right]$$
$$\times\left\{\pm 16z+\sqrt{1-\alpha^2 z}\left[\sqrt{1+(2-\alpha)\sqrt{z}}\,\sqrt{1+(2+\alpha)\sqrt{z}}\right.\right.$$
$$\left.\left.+\sqrt{1-(2-\alpha)\sqrt{z}}\,\sqrt{1-(2+\alpha)\sqrt{z}}\right]^2\right\} \tag{4.19}$$

and $z = 1/w^2$. This product form is of particular importance because it can be used to determine the Green function $G(2n,n,n;\alpha,w)$ for all $\alpha\in(0,\infty)$ at *any* point $w\in\mathcal{C}^-$. When $n=0$, formula (4.18) reduces to a known result for $G(0,0,0;\alpha,w)$ (see Delves and Joyce (2001a), equation (5.13)).

Finally, we note that the application of a contiguous ${}_2F_1$ relation to (4.16) enables one to express the product form (4.18) in terms of complete elliptic integrals. In particular, it is found that

$$wG(2n,n,n;\alpha,w) = \left(\frac{8}{\pi^2}\right)\frac{\left(\frac{1}{4}\right)_n}{\left(\frac{3}{4}\right)_n}\frac{1}{(2\alpha z)^{2n}}\left(\sqrt{1-(2-\alpha)^2 z}+\sqrt{1-(2+\alpha)^2 z}\right)^{4n-1}$$
$$\times\left[A_n^{(0)}(k_+)K(k_+)-A_n^{(1)}(k_+)E(k_+)\right]$$
$$\times\left[A_n^{(0)}(k_-)K(k_-)-A_n^{(1)}(k_-)E(k_-)\right] \tag{4.20}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds, respectively, with a modulus k. The coefficients $\{A_n^{(j)}(k) : j = 0, 1\}$ are polynomials in the variable k which can be determined using the recurrence relation

$$4(4n+1)A_{n+1}^{(j)}(k) - 4n\left(1+4k^2-4k^4\right)A_n^{(j)}(k) + (4n-1)k^2\left(1-k^2\right)A_{n-1}^{(j)}(k) = 0 \tag{4.21}$$

where $j = 0, 1$ and $n = 1, 2, \ldots$. The initial conditions for (4.21) are $A_0^{(0)}(k) = 1$, $A_1^{(0)}(k) = \frac{1}{2}\left(1-k^2\right)$ and $A_0^{(1)}(k) = 0$, $A_1^{(1)}(k) = \frac{1}{2}\left(1-2k^2\right)$.

4.3. ${}_2F_1$ product formulae for $G^-(2n, n, n; \alpha, w_1)$

We now make the substitution $w = w_1 - \mathrm{i}\epsilon$ into (4.11), where w_1 is real and $\epsilon > 0$, and then apply the definition (1.4). Hence, we find that

$$\begin{aligned} G^-(2n, n, n; \alpha, w_1) = {} & \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}\mathrm{i}\left(\alpha^2-4-w_1^2\right)^{-1/2} \\ & \times\left[\frac{w_1^2}{8\alpha}\left(\sqrt{\frac{(2-\alpha)^2}{w_1^2}-1}-\sqrt{\frac{(2+\alpha)^2}{w_1^2}-1}\right)^2\right]^{2n} \\ & \times {}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\widetilde{\eta}_+\right){}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\widetilde{\eta}_-\right) \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \widetilde{\eta}_\pm \equiv \widetilde{\eta}_\pm(\alpha, w_1) = {} & \lim_{\epsilon\to 0+}\eta_\pm(\alpha, w_1-\mathrm{i}\epsilon) \\ = {} & \frac{1}{2} - \frac{w_1^2}{2\left(w_1^2+4-\alpha^2\right)^2} \\ & \times\left[\pm 16\mathrm{i}\sqrt{\frac{\alpha^2}{w_1^2}-1}+\left(4+\alpha^2-w_1^2\right)\sqrt{\frac{(2-\alpha)^2}{w_1^2}-1}\sqrt{\frac{(2+\alpha)^2}{w_1^2}-1}\right]. \end{aligned} \tag{4.23}$$

This product form is valid provided that $0 < w_1 < \sqrt{\alpha^2-4}$ and $\alpha \in (2, \infty)$. We can also apply (4.22) when $\sqrt{4+\alpha^2} < w_1 \leqslant 2+\alpha$ and $\alpha \in (0, \infty)$.

In a similar manner, we can use (4.13) to obtain

$$\begin{aligned} G^-\left(2n, n, n; \alpha, w_1\right) = {} & \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}\left(w_1^2+4-\alpha^2\right)^{-1/2}{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\widetilde{\eta}_-\right) \\ & \times\left\{\left[\frac{w_1^2}{8\alpha}\left(\sqrt{\frac{(2-\alpha)^2}{w_1^2}-1}-\sqrt{\frac{(2+\alpha)^2}{w_1^2}-1}\right)^2\right]^{2n}{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\widetilde{\eta}_+\right)\right. \\ & \left.+\mathrm{i}\sqrt{2}\left[-\frac{w_1^2}{\alpha^2}\left(1+\mathrm{i}\sqrt{\frac{\alpha^2}{w_1^2}-1}\right)^2\right]^{n}{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;1-\widetilde{\eta}_+\right)\right\} \end{aligned} \tag{4.24}$$

where $\widetilde{\eta}_\pm \equiv \widetilde{\eta}_\pm(\alpha, w_1)$ are defined in (4.23). This second expression is valid provided that $0 < w_1 \leqslant \sqrt{4+\alpha^2}$ and $\alpha \in (0, 2]$. We can also apply formula (4.24) when $\sqrt{\alpha^2-4} < w_1 \leqslant \sqrt{4+\alpha^2}$ and $\alpha \in (2, \infty)$.

5. Evaluation of $G(2n, n, n;\ \alpha, w)$ and $G^{-}(2n, n, n;\ \alpha, w_1)$ for special values of w and w_1

We shall now show that the product forms for $G(2n, n, n; \alpha, w)$ and $G^{-}(2n, n, n; \alpha, w_1)$ can be simplified when w and w_1 take certain special values.

5.1. Evaluation of $G(2n, n, n; \alpha, 2+\alpha)$

The substitution $w = 2+\alpha$ into (4.11) yields the product form

$$G(2n, n, n; \alpha, 2+\alpha) = \frac{1}{2\sqrt{2+\alpha}} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^2} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \frac{1}{2} + \frac{\sqrt{1+\alpha}}{2+\alpha}\right) \times {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \frac{1}{2} - \frac{\sqrt{1+\alpha}}{2+\alpha}\right) \tag{5.1}$$

where $\alpha \in (0, \infty)$. This result is particularly useful for investigating the asymptotic behaviour of $G(2n, n, n; \alpha, 2+\alpha)$ as $n \to \infty$.

When $w = 2+\alpha$, it is also possible to use ${}_2F_1$ transformation formulae to write (4.18) in the much simplified form

$$G(2n, n, n; \alpha, 2+\alpha) = \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{4(n!)^2} \left(1 - 2k_1 - k_1^2\right)\left[k_1\left(1 - k_1^2\right)\right]^{2n} \times {}_2F_1\left(n + \frac{1}{2}, 2n + \frac{1}{2}; 2n+1; 1 - k_1^2\right) {}_2F_1\left(n + \frac{1}{2}, 2n + \frac{1}{2}; n+1; k_1^2\right) \tag{5.2}$$

where

$$k_1 \equiv k_1(\alpha) = \frac{1}{\alpha}\left(\sqrt{2}\sqrt{1+\alpha} - \sqrt{2+\alpha}\right)\left(\sqrt{2+\alpha} - \sqrt{2}\right) \tag{5.3}$$

and $\alpha \in (0, \infty)$. It follows from (5.2) that $G(2n, n, n; \alpha, 2+\alpha)$ can be expressed in terms of $K(k_1)$, $E(k_1)$, $K'(k_1)$ and $E'(k_1)$, where $K'(k_1)$ and $E'(k_1)$ are complete elliptic integrals with the complementary modulus $k_1' = \sqrt{1 - k_1^2}$. For example, we find that

$$G(0, 0, 0; \alpha, 2+\alpha) = \frac{1}{\pi^2}\left(1 - 2k_1 - k_1^2\right) K'(k_1) K(k_1) \tag{5.4}$$

$$G(2, 1, 1; \alpha, 2+\alpha) = \frac{1}{3\pi^2} \frac{\left(1 - 2k_1 - k_1^2\right)}{k_1^2\left(1 - k_1^2\right)^2} \left[\left(1 - k_1^2\right)K(k_1) - \left(1 + k_1^2\right)E(k_1)\right] \times \left[2k_1^2 K'(k_1) - \left(1 + k_1^2\right)E'(k_1)\right] \tag{5.5}$$

$$G(4, 2, 2; \alpha, 2+\alpha) = \frac{1}{105\pi^2} \frac{\left(1 - 2k_1 - k_1^2\right)}{k_1^4\left(1 - k_1^2\right)^4} \times \left[\left(1 - k_1^2\right)\left(2 - 9k_1^2 - k_1^4\right)K(k_1) - 2\left(1 + k_1^2\right)\left(1 - 6k_1^2 + k_1^4\right)E(k_1)\right] \times \left[2\left(1 + k_1^2\right)\left(1 - 6k_1^2 + k_1^4\right)E'(k_1) - k_1^2\left(1 - 18k_1^2 + k_1^4\right)K'(k_1)\right]. \tag{5.6}$$

Formula (5.4) was first derived by Delves and Joyce (2001a).

5.2. Evaluation of $G^{-}(2n, n, n; \alpha, \alpha)$

If we make the substitution $w_1 = \alpha$ into (4.24), we obtain the simplified product form

$$G^{-}(2n,n,n;\alpha,\alpha) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2(n!)^2}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}-\frac{1}{2}\sqrt{1-\alpha^2}\right) \times\left\{\left[\frac{1}{2\alpha}\left(\sqrt{1+\alpha}-\sqrt{1-\alpha}\right)^2\right]^{2n}{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}-\frac{1}{2}\sqrt{1-\alpha^2}\right) + \mathrm{i}\sqrt{2}(-1)^n\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}+\frac{1}{2}\sqrt{1-\alpha^2}\right)\right\} \tag{5.7}$$

where $\alpha \in (0,\infty)$. For the special case $\alpha = 1$, this formula reduces to a known result for $G^{-}(2n, n, n; 1, 1)$ (see Joyce and Delves (2004b), equations (7.9)–(7.11)).

It is also possible to use ${}_2F_1$ transformation formulae to write (5.7) in the form

$$G^{-}(2n,n,n;\alpha,\alpha) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2(n!)^2}\left(1+k_2^2\right)\left[2k_2\left(1-k_2^2\right)\right]^{2n} \times{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};n+1;k_2^2\right)\left[{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};n+1;k_2^2\right) + \mathrm{i}\frac{(-1)^n}{2^{2n}}\,{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};2n+1;1-k_2^2\right)\right] \tag{5.8}$$

where

$$k_2 \equiv k_2(\alpha) = \frac{1}{\alpha^2}\left(\sqrt{2}\sqrt{1-\sqrt{1-\alpha^2}}-\alpha\right)\left(1+\sqrt{1-\alpha^2}\right) \tag{5.9}$$

and $\alpha \in (0,\infty)$. It follows from (5.8) that $G^{-}(2n, n, n; \alpha, \alpha)$ can be evaluated in terms of $K(k_2)$, $E(k_2)$, $K'(k_2)$ and $E'(k_2)$. For example, we find that

$$G^{-}(0,0,0;\alpha,\alpha) = \frac{2}{\pi^2}\left(1+k_2^2\right)K(k_2)\left[K(k_2)+\mathrm{i}K'(k_2)\right] \tag{5.10}$$

with $\alpha \in (0,\infty)$. It should be noted that the modulus $k_2 \equiv k_2(\alpha)$ is a *complex* number when $\alpha > 1$.

5.3. Evaluation of $G^{-}(2n, n, n; \alpha, |2-\alpha|)$

Next, we assume that $\alpha \in (0, 2)$ and make the substitution $w_1 = 2 - \alpha$ into (4.24). This procedure yields

$$G^{-}(2n,n,n;\alpha,2-\alpha) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2(n!)^2}\frac{1}{\sqrt{2-\alpha}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}+\mathrm{i}\frac{\sqrt{\alpha-1}}{2-\alpha}\right) \times\left\{{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}-\mathrm{i}\frac{\sqrt{\alpha-1}}{2-\alpha}\right)+\mathrm{i}\sqrt{2}\left[-\frac{1}{\alpha^2}\left(2-\alpha+2\mathrm{i}\sqrt{\alpha-1}\right)^2\right]^n \times{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}+\mathrm{i}\frac{\sqrt{\alpha-1}}{2-\alpha}\right)\right\} \tag{5.11}$$

where $\alpha \in (0, 2)$. When $\alpha \in (2,\infty)$, we find from (4.22) that

$$G^{-}(2n,n,n;\alpha,\alpha-2) = \mathrm{i}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2(n!)^2}\frac{1}{\sqrt{\alpha-2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}+\mathrm{i}\frac{\sqrt{\alpha-1}}{2-\alpha}\right) \times{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{1}{2}-\mathrm{i}\frac{\sqrt{\alpha-1}}{2-\alpha}\right). \tag{5.12}$$

We can also use ${}_2F_1$ transformation formulae to express (5.11) in the form

$$G^-(2n,n,n;\alpha,2-\alpha) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{4(n!)^2}\left(1+2k_3-k_3^2\right)\left[k_3\left(1-k_3^2\right)\right]^{2n} \times {}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};n+1;k_3^2\right)\left[{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};2n+1;1-k_3^2\right) + 2\mathrm{i}(-4)^n\,{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};n+1;k_3^2\right)\right] \tag{5.13}$$

where

$$k_3 \equiv k_3(\alpha) = \frac{\mathrm{i}}{\alpha}\left(\sqrt{2}\sqrt{\alpha-1}-\sqrt{\alpha-2}\right)\left(\sqrt{2}+\mathrm{i}\sqrt{\alpha-2}\right) \tag{5.14}$$

and $\alpha \in (0,2]$. If the same transformation procedure is applied to (5.12), we obtain

$$G^-(2n,n,n;\alpha,\alpha-2) = \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{4(n!)^2}\left(1+2k_3-k_3^2\right)\left[k_3\left(1-k_3^2\right)\right]^{2n} \times {}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};2n+1;1-k_3^2\right){}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};n+1;k_3^2\right) \tag{5.15}$$

where $\alpha \in (2,\infty)$. It follows from (5.13) and (5.15) that $G^-(2n,n,n;\alpha,|2-\alpha|)$ can be evaluated in terms of $K(k_3)$, $E(k_3)$, $K'(k_3)$ and $E'(k_3)$ for all $\alpha \in (0,\infty)$. For example, we find that

$$G^-(0,0,0;\alpha,2-\alpha) = \frac{1}{\pi^2}\left(1+2k_3-k_3^2\right)K(k_3)\left[K'(k_3)+2\mathrm{i}K(k_3)\right] \tag{5.16}$$

where $\alpha \in (0,2]$. When $\alpha \in (2,\infty)$, we have the simpler formula

$$G^-(0,0,0;\alpha,\alpha-2) = \frac{1}{\pi^2}\left(1+2k_3-k_3^2\right)K'(k_3)K(k_3). \tag{5.17}$$

Finally, we note that $G_{\mathrm{R}}(2n,n,n;\alpha,\alpha-2) \equiv 0$ for all $\alpha \in [2,\infty)$.

5.4. Evaluation of $G^-(2n,n,n;\alpha,0)$

For this special case, we first assume that $\alpha \in (0,2)$ and then take the limit $w_2 \to 0-$ in (4.15). This procedure gives

$$G^-(2n,n,n;\alpha,0) = \mathrm{i}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2}\sqrt{\frac{2}{4-\alpha^2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;-\frac{\alpha^2}{4-\alpha^2}\right) \times \mathrm{Re}\left[{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;\frac{4}{4-\alpha^2}\right)\right] \tag{5.18}$$

where $\alpha \in (0,2)$. When $\alpha \in (2,\infty)$, we can make the substitution $w = \mathrm{i}w_2$ into (4.11), where $w_2 < 0$, and then take the limit $w_2 \to 0-$. Hence, we obtain

$$G^-(2n,n,n;\alpha,0) = \mathrm{i}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^2\sqrt{\alpha^2-4}}\left(\frac{2}{\alpha}\right)^{2n}\left[{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1;-\frac{4}{\alpha^2-4}\right)\right]^2. \tag{5.19}$$

Next, we apply the same limiting procedure to formula (4.18). In this manner, it is found that

$$G^-(2n,n,n;\alpha,0) = \mathrm{i}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{2(n!)^2}\left(\frac{\alpha}{2}\right)^{2n} \times {}_2F_1\left(\frac{1}{2},2n+\frac{1}{2};n+1;1-k_4^2\right){}_2F_1\left(\frac{1}{2},2n+\frac{1}{2};n+1;k_4^2\right) \tag{5.20}$$

where

$$k_4 \equiv k_4(\alpha) = \frac{1}{2\sqrt{2}}\left(\sqrt{2+\alpha} - \sqrt{2-\alpha}\right) \tag{5.21}$$

and $\alpha \in (0, 2]$. When $\alpha \in [2, \infty)$, we have the further result

$$G^-(2n, n, n; \alpha, 0) = \mathrm{i}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{\alpha(n!)^2}\left(\frac{2}{\alpha}\right)^{2n}\left[{}_2F_1\left(\frac{1}{2}, 2n+\frac{1}{2}; n+1; k_5^2\right)\right]^2 \tag{5.22}$$

where

$$k_5 \equiv k_5(\alpha) = \frac{1}{2\sqrt{\alpha}}\left(\sqrt{2+\alpha} - \sqrt{\alpha-2}\right). \tag{5.23}$$

If $\alpha = 2$, we can use (5.22) and the identity (Erdélyi *et al* 1953, p 104, equation (50))

$${}_2F_1\left(\frac{1}{2}, 2n+\frac{1}{2}; n+1; \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\left[\Gamma\left(\frac{3}{4}\right)\right]^2}\frac{n!}{\left(\frac{3}{4}\right)_n} \tag{5.24}$$

to obtain the formula

$$G^-(2n, n, n; 2, 0) = \mathrm{i}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}{8\pi^3}\frac{\left(\frac{1}{4}\right)_n}{\left(\frac{3}{4}\right)_n} = \mathrm{i}\frac{2}{\pi^2}\{K(k[1])\}^2\frac{\left(\frac{1}{4}\right)_n}{\left(\frac{3}{4}\right)_n} \tag{5.25}$$

where $k[1] = 1/\sqrt{2}$ is the *singular value* of order 1 (see Borwein and Borwein (1987), p 139).

It is possible to express (5.20) in terms of complete elliptic integrals using (4.20). The final result is

$$G^-(2n, n, n; \alpha, 0) = \mathrm{i}\left(\frac{2}{\pi^2}\right)\frac{\left(\frac{1}{4}\right)_n}{\left(\frac{3}{4}\right)_n}\left(\frac{8}{\alpha}\right)^{2n}\left[A_n^{(0)}(k_4')K'(k_4) - A_n^{(1)}(k_4')E'(k_4)\right]$$
$$\times\left[A_n^{(0)}(k_4)K(k_4) - A_n^{(1)}(k_4)E(k_4)\right] \tag{5.26}$$

where $\alpha \in (0, 2]$ and $\left\{A_n^{(j)}(k) : j = 0, 1\right\}$ satisfy the recurrence relation (4.21). When $\alpha \in [2, \infty)$, we find that (4.20) gives

$$G^-(2n, n, n; \alpha, 0) = \mathrm{i}\left(\frac{4}{\alpha\pi^2}\right)\frac{\left(\frac{1}{4}\right)_n}{\left(\frac{3}{4}\right)_n}(2\alpha)^{2n}\left[A_n^{(0)}(k_5)K(k_5) - A_n^{(1)}(k_5)E(k_5)\right]^2. \tag{5.27}$$

5.5. Evaluation of $G^-\left(2n, n, n; \alpha, \sqrt{\alpha^2-4}\right)$ and $G\left(2n, n, n; \alpha, \pm\mathrm{i}\sqrt{4-\alpha^2}\right)$

The point $w = \sqrt{\alpha^2-4}$, with $\alpha \in (2, \infty)$, is of interest because it belongs to the boundary set $\mathcal{S}$ which was discussed in section 4. If we substitute $w = \sqrt{\alpha^2-4} - \mathrm{i}\epsilon$ into (4.11) and then take the limit $\epsilon \to 0+$, it is eventually found that

$$G^-\left(2n, n, n; \alpha, \sqrt{\alpha^2-4}\right) = \mathrm{i}\frac{\exp(-\mathrm{i}\pi/8)}{\pi 2^{1/4}(\alpha^2-4)^{1/8}}K(k[1])\frac{\left(\frac{1}{4}\right)_n}{n!}$$
$$\times\left[\frac{1}{2\alpha}\left(\sqrt{2+\alpha} - \mathrm{i}\sqrt{\alpha-2}\right)^2\right]^{2n}{}_2F_1\left[\frac{1}{4}, \frac{3}{4}; n+1; \frac{1}{2} + \mathrm{i}\frac{(8-\alpha^2)}{8\sqrt{\alpha^2-4}}\right] \tag{5.28}$$

where $k[1] = 1/\sqrt{2}$ and $\alpha \in (2, \infty)$. In the limit $\alpha \to 2+$, formula (5.28) reduces to the expected result (5.25).

A similar analysis can also be carried out for the boundary points $w = \pm \mathrm{i}\sqrt{4-\alpha^2}$, with $\alpha \in (0, 2)$. In particular, we find that

$$G\Big(2n, n, n; \alpha, \pm \mathrm{i}\sqrt{4-\alpha^2}\Big) = \mp \mathrm{i} \frac{K(k[1])}{\pi 2^{1/4}\big(4-\alpha^2\big)^{1/8}} \frac{\big(\frac{1}{4}\big)_n}{n!} \times \left[\frac{1}{2\alpha}\Big(\sqrt{2+\alpha} - \sqrt{2-\alpha}\Big)^2\right]^{2n} {}_2F_1\left[\frac{1}{4}, \frac{3}{4}; n+1; \frac{1}{2} - \frac{\big(8-\alpha^2\big)}{8\sqrt{4-\alpha^2}}\right]. \tag{5.29}$$

In section 6, we shall find that (5.28) and (5.29) play an important role in the analysis of the asymptotic behaviour of $G(2n, n, n; \alpha, w)$ as $n \to \infty$.

6. Asymptotic behaviour of $G(2n, n, n; \alpha, w)$ and $G^-\big(2n, n, n; \alpha, w_1\big)$ as $n \to \infty$

Our main aim in this section is to show that the ${}_2F_1$ product forms (4.11), (4.13) and (4.15) enable one to derive uniform asymptotic expansions for $G(2n, n, n; \alpha, w)$, as $n \to \infty$, in a very *direct* and *simple* manner.

6.1. General asymptotic representations

We begin by considering the standard asymptotic formula (Luke 1969, p 235)

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; n+1; \eta\right) \sim \Lambda_M(n, \eta) \tag{6.1}$$

as $n \to \infty$, where

$$\Lambda_M(n, \eta) \equiv \sum_{m=0}^{M} \frac{\big(\frac{1}{4}\big)_m \big(\frac{3}{4}\big)_m}{(n+1)_m m!} \eta^m \tag{6.2}$$

and $M = 0, 1, 2, \ldots$. Next, we apply (6.1) to the product form (4.11). This procedure yields the asymptotic representation

$$wG(2n, n, n; \alpha, w) \sim \frac{1}{\pi\sqrt{2}} \frac{\Gamma\big(n+\frac{1}{4}\big)\Gamma\big(n+\frac{3}{4}\big)}{[\Gamma(n+1)]^2} \left(\frac{w^2}{w^2+4-\alpha^2}\right)^{1/2} \times \left[\frac{w^2}{8\alpha}\left(\sqrt{1-\frac{(2-\alpha)^2}{w^2}} - \sqrt{1-\frac{(2+\alpha)^2}{w^2}}\right)^2\right]^{2n} \Lambda_M\big(n, \eta_+\big)\Lambda_M\big(n, \eta_-\big) \tag{6.3}$$

as $n \to \infty$, where M is *fixed* and $\eta_\pm \equiv \eta_\pm(\alpha, w)$ are defined in (4.12). We expect (6.3) to be valid provided that w lies in the outer region $\mathcal{D}(\alpha)$ of the cut w plane.

A uniform asymptotic expansion for $G(2n, n, n; \alpha, w)$ can now be derived by expanding the ratio of gamma functions and the Λ functions in (6.3) in powers of $1/n$. In particular, we find that

$$wG(2n, n, n; \alpha, w) \sim \frac{1}{\pi n\sqrt{2}} \left[\frac{w^2}{8\alpha}\left(\sqrt{1-\frac{(2-\alpha)^2}{w^2}} - \sqrt{1-\frac{(2+\alpha)^2}{w^2}}\right)^2\right]^{2n} \times \left(\frac{w^2}{w^2+4-\alpha^2}\right)^{1/2} \sum_{m=0}^{\infty} \frac{b_m^{(1)}(\alpha, w)}{n^m} \tag{6.4}$$

as $n \to \infty$, where $w \in \mathcal{C}^-$ lies in the region $\mathcal{D}(\alpha)$ and $b_0^{(1)}(\alpha, w) = 1$. In appendix A, we give formulae for the higher-order coefficients $\{b_m^{(1)}(\alpha, w) : m = 1, 2, 3, 4\}$.

In a similar manner, we can apply (6.1) to (4.13). Hence, we obtain

$$\begin{aligned} wG(2n, n, n; \alpha, w) &\sim \frac{1}{\pi n\sqrt{2}} \left(\frac{w^2}{w^2 + 4 - \alpha^2}\right)^{1/2} \\ &\quad \times \left\{ \left[\frac{w^2}{8\alpha} \left(\sqrt{1 - \frac{(2-\alpha)^2}{w^2}} - \sqrt{1 - \frac{(2+\alpha)^2}{w^2}} \right)^2 \right]^{2n} \sum_{m=0}^{\infty} \frac{b_m^{(1)}(\alpha, w)}{n^m} \right. \\ &\quad \left. \pm \mathrm{i}\sqrt{2} \left[-\frac{w^2}{\alpha^2} \left(1 - \sqrt{1 - \frac{\alpha^2}{w^2}} \right)^2 \right]^n \sum_{m=0}^{\infty} \frac{b_m^{(2)}(\alpha, w)}{n^m} \right\} \end{aligned} \tag{6.5}$$

as $n \to \infty$, where $b_0^{(2)}(\alpha, w) = 1$. This representation is valid when $\alpha \in [2, \infty)$ and $w \in \mathcal{C}^-$ lies in one of the inner regions $\mathcal{D}_1^{\pm}(\alpha)$. It is also valid when $\alpha \in (0, 2)$ and $w \in \mathcal{C}^-$ lies in the region $\mathcal{D}_1(\alpha)$, with $\mathrm{Re}(w) \neq 0$. The role of the $\pm$ signs in (6.5) is the same as in equation (4.13). Formulae for the coefficients $\{b_m^{(2)}(\alpha, w) : m = 1, 2, 3, 4\}$ are given in appendix B. It should be noted that the coefficients $\{b_m^{(j)}(\alpha, w) : j = 1, 2; m = 1, 2, \ldots\}$ in the expansions (6.4) and (6.5) all become infinite as $w^2 \to (\alpha^2 - 4)$.

When $\alpha \in (0, 2)$ and $w \in \mathcal{C}^-$ lies in the region $\mathcal{D}_1(\alpha)$, with $\mathrm{Re}(w) = 0$, we can use (4.15) to derive the alternative expansion

$$G\big(2n, n, n; \alpha, \mathrm{i}w_2\big) \sim \pm\frac{\mathrm{i}}{\pi n} \left(\frac{1}{4 - \alpha^2 - w_2^2}\right)^{1/2} \left[\frac{w_2^2}{\alpha^2} \left(1 - \sqrt{1 + \frac{\alpha^2}{w_2^2}} \right)^2 \right]^n \sum_{m=0}^{\infty} \frac{b_m^{(2)}(\alpha, \mathrm{i}w_2)}{n^m} \tag{6.6}$$

as $n \to \infty$. The upper positive sign in (6.6) is valid for $-\sqrt{4 - \alpha^2} < w_2 < 0$, while the lower negative sign is valid for $0 < w_2 < \sqrt{4 - \alpha^2}$.

Next, we let $w = w_1 - \mathrm{i}\epsilon$ in (6.4) and then take the limit $\epsilon \to 0+$. This procedure yields

$$\begin{aligned} G^-\big(2n, n, n; \alpha, w_1\big) &\sim \frac{1}{\pi n\sqrt{2}} \left[\frac{w_1^2}{8\alpha} \left(\sqrt{\frac{(2-\alpha)^2}{w_1^2} - 1} - \sqrt{\frac{(2+\alpha)^2}{w_1^2} - 1} \right)^2 \right]^{2n} \\ &\quad \times \mathrm{i}\big(\alpha^2 - 4 - w_1^2\big)^{-1/2} \sum_{m=0}^{\infty} \frac{\widetilde{b}_m^{(1)}\big(\alpha, w_1\big)}{n^m} \end{aligned} \tag{6.7}$$

as $n \to \infty$, where $\widetilde{b}_0^{(1)}\big(\alpha, w_1\big) = 1$. This result is valid provided that $0 < w_1 < \sqrt{\alpha^2 - 4}$ and $\alpha \in (2, \infty)$. We can also apply (6.7) when $\sqrt{4 + \alpha^2} < w_1 \leqslant 2 + \alpha$ and $\alpha \in (0, \infty)$. Formulae for $\{\widetilde{b}_m^{(1)}\big(\alpha, w_1\big) : m = 1, 2, 3, 4\}$ are readily obtained by making the formal replacements $w \mapsto w_1$ and

$$\sqrt{1 - \frac{(2 \pm \alpha)^2}{w^2}} \mapsto -\mathrm{i}\sqrt{\frac{(2 \pm \alpha)^2}{w_1^2} - 1} \tag{6.8}$$

in the right-hand side of the expressions listed in appendix A.

Finally, we substitute $w = w_1 - \mathrm{i}\epsilon$ into (6.5) and then take the limit $\epsilon \to 0+$. Hence, we obtain the further asymptotic expansion

$$G^{-}\big(2n,n,n;\alpha,w_1\big)\sim\frac{1}{\pi n\sqrt{2}}\left(\frac{1}{w_1^2+4-\alpha^2}\right)^{1/2}$$
$$\times\left\{\left[\frac{w_1^2}{8\alpha}\left(\sqrt{\frac{(2-\alpha)^2}{w_1^2}-1}-\sqrt{\frac{(2+\alpha)^2}{w_1^2}-1}\right)^2\right]^{2n}\sum_{m=0}^{\infty}\frac{\widetilde{b}_m^{(1)}\big(\alpha,w_1\big)}{n^m}\right.$$
$$\left.+\mathrm{i}\sqrt{2}\left[-\frac{w_1^2}{\alpha^2}\left(1+\mathrm{i}\sqrt{\frac{\alpha^2}{w_1^2}-1}\right)^2\right]^{n}\sum_{m=0}^{\infty}\frac{\widetilde{b}_m^{(2)}\big(\alpha,w_1\big)}{n^m}\right\} \tag{6.9}$$

as $n\to\infty$, where $\widetilde{b}_0^{(2)}\big(\alpha,w_1\big)=1$. This result is valid provided that $0<w_1\leqslant\sqrt{4+\alpha^2}$ and $\alpha\in(0,2]$. We can also apply (6.9) when $\sqrt{\alpha^2-4}<w_1\leqslant\sqrt{4+\alpha^2}$ and $\alpha\in(2,\infty)$. Formulae for $\big\{\widetilde{b}_m^{(2)}\big(\alpha,w_1\big):m=1,2,3,4\big\}$ can be written down by making the formal replacements $w\mapsto w_1$ and

$$\sqrt{1-\frac{\alpha^2}{w^2}}\mapsto-\mathrm{i}\sqrt{\frac{\alpha^2}{w_1^2}-1} \tag{6.10}$$

in the right-hand side of the expressions in appendix B.

6.2. Asymptotic behaviour of $G(2n,n,n;\alpha,2+\alpha)$

The asymptotic behaviour of $G(2n,n,n;\alpha,w)$ for the important special case $w=2+\alpha$ is most conveniently determined by applying the representation (6.1) to the simplified formula (5.1). Hence, we find that

$$G(2n,n,n;\alpha,2+\alpha)\sim\frac{1}{2\pi n\sqrt{2}\,\sqrt{2+\alpha}}\sum_{m=0}^{\infty}\frac{b_{2m}^{(1)}(\alpha,2+\alpha)}{n^{2m}} \tag{6.11}$$

as $n\to\infty$, where $b_0^{(1)}(\alpha,2+\alpha)=1$,

$$b_2^{(1)}(\alpha,2+\alpha)=-\frac{3\alpha^2}{32(2+\alpha)^2} \tag{6.12}$$

$$b_4^{(1)}(\alpha,2+\alpha)=-\frac{3\alpha^2}{2\,048(2+\alpha)^4}\big(256+256\alpha-41\alpha^2\big) \tag{6.13}$$

$$b_6^{(1)}(\alpha,2+\alpha)=-\frac{3\alpha^2}{65\,536(2+\alpha)^6}\big(32\,768+65\,536\alpha-18\,048\alpha^2-50\,816\alpha^3+2\,573\alpha^4\big) \tag{6.14}$$

$$b_8^{(1)}(\alpha,2+\alpha)=-\frac{3\alpha^2}{8\,388\,608(2+\alpha)^8}\big(16\,777\,216+50\,331\,648\alpha-81\,592\,320\alpha^2$$
$$-247\,070\,720\alpha^3-76\,846\,080\alpha^4+55\,077\,888\alpha^5-1\,371\,761\alpha^6\big) \tag{6.15}$$

and $\alpha\in(0,\infty)$. When $\alpha=1$, the expansion (6.11) is consistent with the work of Duffin (1953) and Joyce and Delves (2004b).

6.3. Asymptotic behaviour of $G^{-}(2n,n,n;\alpha,0)$

For this case, we begin by assuming that $\alpha\in(0,2)$ and then apply the basic representation (6.1) to the product form (5.18). This procedure gives

$$G^{-}(2n,n,n;\alpha,0)\sim\frac{\mathrm{i}}{\pi n\sqrt{4-\alpha^2}}\sum_{m=0}^{\infty}\frac{c_{2m}^{(2)}(\alpha)}{n^{2m}} \tag{6.16}$$

as $n \to \infty$, where $c_0^{(2)}(\alpha) = 1$,

$$c_2^{(2)}(\alpha) = \frac{3\alpha^2}{2\left(4-\alpha^2\right)^2} \tag{6.17}$$

$$c_4^{(2)}(\alpha) = \frac{3\alpha^2}{8\left(4-\alpha^2\right)^4}\left(64 + 73\alpha^2 + 4\alpha^4\right) \tag{6.18}$$

$$c_6^{(2)}(\alpha) = \frac{3\alpha^2}{16\left(4-\alpha^2\right)^6}\left(2\,048 + 14\,752\alpha^2 + 9\,693\alpha^4 + 922\alpha^6 + 8\alpha^8\right) \tag{6.19}$$

$$c_8^{(2)}(\alpha) = \frac{3\alpha^2}{128\left(4-\alpha^2\right)^8}\left(262\,144 + 8\,638\,464\alpha^2 + 22\,260\,480\alpha^4 + 11\,430\,385\alpha^6 + 1\,391\,280\alpha^8 + 33\,744\alpha^{10} + 64\alpha^{12}\right) \tag{6.20}$$

and $\alpha \in (0, 2)$. It is found that $\left\{c_{2m}^{(2)}(\alpha) = b_{2m}^{(2)}(\alpha, 0) : m = 0, 1, 2, \ldots\right\}$.

When $\alpha \in (2, \infty)$, we can use formula (5.19) to obtain the alternative expansion

$$G^-(2n, n, n; \alpha, 0) \sim \frac{\mathrm{i}}{\pi n\sqrt{2}\,\sqrt{\alpha^2-4}}\left(\frac{2}{\alpha}\right)^{2n}\sum_{m=0}^{\infty}\frac{c_m^{(1)}(\alpha)}{n^m} \tag{6.21}$$

as $n \to \infty$, where $c_0^{(1)}(\alpha) = 1$,

$$c_1^{(1)}(\alpha) = -\frac{3\left(4+\alpha^2\right)}{16(\alpha^2-4)} \tag{6.22}$$

$$c_2^{(1)}(\alpha) = \frac{3}{512(\alpha^2-4)^2}\left(48 + 280\alpha^2 + 3\alpha^4\right) \tag{6.23}$$

$$c_3^{(1)}(\alpha) = \frac{3\left(4+\alpha^2\right)}{8\,192(\alpha^2-4)^3}\left(208 - 5\,144\alpha^2 + 13\alpha^4\right) \tag{6.24}$$

$$c_4^{(1)}(\alpha) = -\frac{3}{524\,288(\alpha^2-4)^4}\left(46\,848 - 5\,081\,344\alpha^2 - 5\,233\,504\alpha^4 - 317\,584\alpha^6 + 183\alpha^8\right). \tag{6.25}$$

It is found that $\left\{c_{2j}^{(1)}(\alpha) = b_{2j}^{(1)}(\alpha, 0) : j = 0, 1, 2, \ldots\right\}$ and

$$\left\{c_{2j+1}^{(1)}(\alpha) = \lim_{w\to 0} b_{2j+1}^{(1)}(\alpha, w) : j = 0, 1, 2, \ldots\right\}. \tag{6.26}$$

The expansions (6.16) and (6.21) are clearly not valid when $\alpha = 2$. In order to deal with this special case, we first consider the standard expansion (Luke 1969, p 34)

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim (z+a-\rho)^{a-b}\sum_{m=0}^{\infty}\frac{(b-a)_{2m}B_{2m}^{(2\rho)}(\rho)}{(2m)!(z+a-\rho)^{2m}} \tag{6.27}$$

as $z \to \infty$, where $B_{2m}^{(2\rho)}(\rho)$ is a generalized Bernoulli polynomial and

$$\rho = \frac{1}{2}(1+a-b). \tag{6.28}$$

If we now apply (6.27) to (5.25), we obtain the required asymptotic expansion

$$G^-(2n, n, n; 2, 0) \sim \mathrm{i}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\pi^2\sqrt{2n}}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{2m}B_{2m}^{(\frac{1}{2})}\left(\frac{1}{4}\right)}{(2m)!\,n^{2m}} \tag{6.29}$$

as $n \to \infty$. From this result, it follows that

$$G^{-}(2n, n, n; 2, 0) \sim \mathrm{i}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\pi^2\sqrt{2n}}\left(1 - \frac{1}{64n^2} + \frac{21}{8\,192n^4} - \frac{671}{524\,288n^6} + \frac{180\,323}{134\,217\,728n^8} - \frac{20\,898\,423}{8\,589\,934\,592n^{10}} + \frac{7\,426\,362\,705}{1\,099\,511\,627\,776n^{12}} + \cdots\right) \tag{6.30}$$

as $n \to \infty$. A striking feature of this expansion is that the amplitude factor $n^{-1/2}$ does not obey the expected n^{-1} decay law.

We can investigate why the expansions (6.16) and (6.21) break down when $\alpha = 2$ by applying the transformation $z = 1/w^2$ to the independent variable in the differential equation (3.4). It is found that if $\alpha = 2$ the transformed equation has a *turning point* of multiplicity 4 at $w = 0$ (Olver 1977).

6.4. Asymptotic behaviour of $G^{-}\left(2n, n, n; \alpha, \sqrt{\alpha^2 - 4}\right)$

It is clear that the asymptotic expansion (6.7) is not valid when $w = \sqrt{\alpha^2 - 4}$ and $\alpha \in (2, \infty)$. For this special case, we can use (5.28) to obtain the alternative result

$$G^{-}\left(2n, n, n; \alpha, \sqrt{\alpha^2 - 4}\right) \sim \mathrm{i}\frac{\Gamma\left(\frac{1}{4}\right)\exp(-\mathrm{i}\pi/8)}{(2\pi)^{3/2}(\alpha^2 - 4)^{1/8}(2n)^{3/4}} \times \left[\frac{1}{2\alpha}\left(\sqrt{2 + \alpha} - \mathrm{i}\sqrt{\alpha - 2}\right)^2\right]^{2n} \sum_{m=0}^{\infty} \frac{d_m(\alpha)}{n^m} \tag{6.31}$$

as $n \to \infty$, where $d_0(\alpha) = 1$,

$$d_1(\alpha) = \frac{3\mathrm{i}}{128(\alpha^2 - 4)^{1/2}}\left(8 - \alpha^2\right) \tag{6.32}$$

$$d_2(\alpha) = \frac{7}{32\,768(\alpha^2 - 4)}\left(8 + 3\alpha^2\right)\left(8 - 5\alpha^2\right) \tag{6.33}$$

$$d_3(\alpha) = \frac{231\mathrm{i}}{4\,194\,304(\alpha^2 - 4)^{3/2}}\left(8 - \alpha^2\right)\left(8 + 3\alpha^2\right)\left(8 - 5\alpha^2\right) \tag{6.34}$$

$$d_4(\alpha) = \frac{77}{2\,147\,483\,648(\alpha^2 - 4)^2}\left(421\,888 - 210\,944\alpha^2 + 381\,568\alpha^4 - 88\,800\alpha^6 + 8\,775\alpha^8\right) \tag{6.35}$$

and $\alpha \in (2, \infty)$.

The expansion (6.7) breaks down at $w = \sqrt{\alpha^2 - 4}$ because the differential equation (3.4) has a *turning point* of multiplicity 2 at $z = 1/(\alpha^2 - 4)$. Finally, we note that it is also possible to use (5.29) to determine the asymptotic behaviour of $G\left(2n, n, n; \alpha, \pm\mathrm{i}\sqrt{4 - \alpha^2}\right)$ as $n \to \infty$, where $\alpha \in (0, 2)$.

7. Concluding remarks

It is possible to establish a recurrence relation for $G(2n, n, n; \alpha, w)$ by following and extending the methods developed by Iwata (1979). In particular, we find that

$$\begin{aligned}
&\alpha^4(n-1)(4n+5)(4n+7)G_{n+2}\\
&-4\alpha^2(n-1)(n+1)^2\big[w^4-2(\alpha^2+12)w^2+(\alpha^2+4)^2\big]G_{n+1}\\
&-2n\Big\{4\big(n^2-1\big)\big[2w^6-\big(5\alpha^2+16\big)w^4+4\big(\alpha^4+2\alpha^2+8\big)w^2\\
&-\alpha^2\big(\alpha^4+4\alpha^2+16\big)\big]+3\alpha^4\Big\}G_n\\
&-4\alpha^2(n+1)(n-1)^2\big[w^4-2(\alpha^2+12)w^2+(\alpha^2+4)^2\big]G_{n-1}\\
&+\alpha^4(n+1)(4n-5)(4n-7)G_{n-2}=0
\end{aligned} \tag{7.1}$$

where $G_n \equiv G(2n,n,n;\alpha,w)$. This recurrence relation leads to alternative methods for investigating the asymptotic behaviour of $G(2n,n,n;\alpha,w)$ as $n\to\infty$. A detailed derivation of (7.1) will be given in a future publication.

Finally, we note that the Green function $G(2n,n,n;\alpha,w)$ can be expressed in the form

$$G(2n,n,n;\alpha,w)=\mathrm{i}J(n;\alpha,w) \tag{7.2}$$

where

$$J(n;\alpha,w)\equiv\int_0^\infty \exp(-\mathrm{i}wt)J_{2n}(\alpha t)J_n^2(t)\,\mathrm{d}t \tag{7.3}$$

and $w\in\mathcal{C}^-$, with $\mathrm{Im}(w)<0$. It follows, therefore, that the results obtained in this paper enable one to write exact formulae and asymptotic expansions for the integral (7.3). For example, if we make the substitution $w=-\mathrm{i}\epsilon$ into (7.2) and take the limit $\epsilon\to 0+$, then we find from (5.20) that

$$\begin{aligned}
J(n;\alpha,0)=\frac{\left(\frac14\right)_n\left(\frac34\right)_n}{2(n!)^2}\left(\frac{\alpha}{2}\right)^{2n}{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12+\frac12\sqrt{1-\frac{\alpha^2}{4}}\right)\\
\times{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12-\frac12\sqrt{1-\frac{\alpha^2}{4}}\right)
\end{aligned} \tag{7.4}$$

where $0<\alpha\leqslant 2$. When $2\leqslant\alpha<\infty$, it is also seen from (5.22) that

$$J(n;\alpha,0)=\frac{\left(\frac14\right)_n\left(\frac34\right)_n}{\alpha(n!)^2}\left(\frac{2}{\alpha}\right)^{2n}\left[{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12-\frac12\sqrt{1-\frac{4}{\alpha^2}}\right)\right]^2. \tag{7.5}$$

It can be proved using various standard ${}_2F_1$ transformation formulae (Erdélyi *et al* 1953) that

$$\begin{aligned}
{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12+\frac12\sqrt{1-x}\right){}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12-\frac12\sqrt{1-x}\right)\\
=x^{-2n-\frac12}\,\mathrm{Re}\left[{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12-\frac12\sqrt{1-\frac1x}\right)\right]^2
\end{aligned} \tag{7.6}$$

where $0<x\leqslant 1$. The application of (7.6) to (7.4), with $x=\alpha^2/4$, yields the *unified* formula

$$J(n;\alpha,0)=\frac{\left(\frac14\right)_n\left(\frac34\right)_n}{\alpha(n!)^2}\left(\frac{2}{\alpha}\right)^{2n}\mathrm{Re}\left[{}_2F_1\left(\frac12,2n+\frac12;n+1;\frac12-\frac12\sqrt{1-\frac{4}{\alpha^2}}\right)\right]^2. \tag{7.7}$$

This new result, which is valid for *all* $\alpha\in(0,\infty)$, provides a partial resolution of a problem that was first discussed by Bailey (1936, p 46).

Appendix A. Coefficients $b_m^{(1)}(\alpha, w)$ for $m = 1, 2, 3, 4$

$$b_1^{(1)}(\alpha, w) = \frac{3w^2(4+\alpha^2-w^2)}{16(w^2+4-\alpha^2)^2}\sqrt{1-\frac{(2-\alpha)^2}{w^2}}\sqrt{1-\frac{(2+\alpha)^2}{w^2}}$$

$$b_2^{(1)}(\alpha, w) = \frac{3}{512(w^2+4-\alpha^2)^4}\Big[(4-\alpha^2)^2(48+280\alpha^2+3\alpha^4) - 4(2\,240+560\alpha^2+268\alpha^4+3\alpha^6)w^2 + 2(2\,192+688\alpha^2+9\alpha^4)w^4 - 4(140+3\alpha^2)w^6 + 3w^8\Big]$$

$$b_3^{(1)}(\alpha, w) = -\frac{3w^2(4+\alpha^2-w^2)}{8192(w^2+4-\alpha^2)^6}\sqrt{1-\frac{(2-\alpha)^2}{w^2}}\sqrt{1-\frac{(2+\alpha)^2}{w^2}}$$

$$\Big[(4-\alpha^2)^2(208-5\,144\alpha^2+13\alpha^4) + 4(41\,152+43\,056\alpha^2+5\,196\alpha^4-13\alpha^6)w^2 - 6(35\,632+4\,304\alpha^2-13\alpha^4)w^4 + 4(2\,572-13\alpha^2)w^6 + 13w^8\Big]$$

$$b_4^{(1)}(\alpha, w) = -\frac{3}{524\,288(w^2+4-\alpha^2)^8}\Big[(4-\alpha^2)^4(46\,848-5\,081\,344\alpha^2 - 5\,233\,504\alpha^4 - 317\,584\alpha^6 + 183\alpha^8) + 8(4-\alpha^2)^2(20\,325\,376+103\,619\,328\alpha^2 + 51\,001\,600\alpha^4 + 8\,457\,792\alpha^6 + 318\,316\alpha^8 - 183\alpha^{10})w^2 - 4(3\,479\,171\,072 + 3\,856\,707\,584\alpha^2 + 479\,105\,280\alpha^4 + 230\,863\,360\alpha^6 + 60\,520\,720\alpha^8 + 2\,154\,672\alpha^{10} - 1\,281\alpha^{12})w^4 + 8(1\,824\,717\,824 + 1\,156\,171\,520\alpha^2 + 355\,551\,360\alpha^4 + 62\,294\,560\alpha^6 + 1\,992\,220\alpha^8 - 1\,281\alpha^{10})w^6 - 10(564\,068\,096 + 278\,889\,472\alpha^2 + 51\,322\,016\alpha^4 + 1\,751\,104\alpha^6 - 1\,281\alpha^8)w^8 + 8(114\,044\,864 + 33\,022\,064\alpha^2 + 1\,431\,324\alpha^4 - 1\,281\alpha^6)w^{10} - 4(13\,590\,512 + 1\,032\,880\alpha^2 - 1\,281\alpha^4)w^{12} + 8(79\,396 - 183\alpha^2)w^{14} + 183w^{16}\Big].$$

Appendix B. Coefficients $b_m^{(2)}(\alpha, w)$ for $m = 1, 2, 3, 4$

$$b_1^{(2)}(\alpha, w) = -\frac{3w^2}{(w^2+4-\alpha^2)^2}\sqrt{1-\frac{\alpha^2}{w^2}}$$

$$b_2^{(2)}(\alpha, w) = \frac{3}{2(w^2+4-\alpha^2)^4}\Big[\alpha^2(4-\alpha^2)^2 - (32+11\alpha^2+4\alpha^4)w^2 + (19+5\alpha^2)w^4 - 2w^6\Big]$$

$$b_3^{(2)}(\alpha, w) = -\frac{3w^2}{4(w^2+4-\alpha^2)^6}\sqrt{1-\frac{\alpha^2}{w^2}}\Big[(4-\alpha^2)^2(64+73\alpha^2+4\alpha^4) - 2(1168+279\alpha^2+114\alpha^4+8\alpha^6)w^2 + 3(338+111\alpha^2+8\alpha^4)w^4 - 2(73+8\alpha^2)w^6 + 4w^8\Big]$$

$$b_4^{(2)}(\alpha, w) = \frac{3}{8\left(w^2+4-\alpha^2\right)^8}\Big[\alpha^2\left(4-\alpha^2\right)^4\left(64+73\alpha^2+4\alpha^4\right)$$
$$-4\left(4-\alpha^2\right)^2\left(512+3\,024\alpha^2+1\,565\alpha^4+246\alpha^6+8\alpha^8\right)w^2$$
$$+\left(201\,728+240\,320\alpha^2+27\,945\alpha^4+14\,980\alpha^6+3\,610\alpha^8+108\alpha^{10}\right)w^4$$
$$-10\left(22\,336+14\,245\alpha^2+4\,434\alpha^4+734\alpha^6+20\alpha^8\right)w^6$$
$$+5\left(17\,461+8\,588\alpha^2+1\,501\alpha^4+44\alpha^6\right)w^8$$
$$-4\left(3\,490+961\alpha^2+36\alpha^4\right)w^{10}+4\left(197+13\alpha^2\right)w^{12}-8w^{14}\Big].$$

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